Noise kernel and the stress energy bitensor of quantum fields in hot flat space and the Schwarzschild black hole under the Gaussian approximation

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We continue our investigation of the properties of the noise kernel in curved spacetimes [N. G. Phillips and B. L. Hu, Phys. Rev. D **63**, 104001 (2001)] by working out explicit examples by means of the modified point-separation scheme and the Gaussian approximation for the Green functions in the manner of Bekenstein, Parker, and Page [J. D. Bekenstein and L. Parker, Phys. Rev. D **23**, 2850 (1981); D. N. Page, *ibid.* **25**, 1499 (1982)]. In the first part we consider the class of optical spacetimes. As a first example we derive the regularized noise kernel for a thermal field in flat space. It is useful for black hole nucleation considerations. In the second example of an optical-Schwarzschild spacetime we obtain after this procedure a finite expression for the noise kernel at the horizon. In the second part we consider the noise kernel for a scalar field in the Schwarzschild black hole. Knowledge of the noise kernel is essential for studying issues related to black hole horizon fluctuations and Hawking radiation back reaction. Much of the work in this part is to determine how the divergent piece conformally transforms under the point-separation scheme. For the Schwarzschild metric we find that the fluctuations of the stress tensor of the Hawking flux in the far field region checks with the analytic results given by Campos and Hu earlier [A. Campos and B. L. Hu, Phys. Rev. D **58**, 125021 (1998); Int. J. Theor. Phys. **38**, 1253 (1999)]. We also verify Page's result for the stress tensor, which, though used often, still lacks a rigorous proof, since in his original work the direct use of the conformal transformation was circumvented. We find that the noise kernel at the Schwarzschild horizon is finite. This dispels speculations in some recent papers that the black hole fluctuations diverge at the horizon. However, as already manifest in the optical case, the Gaussian approximated Green function which works surprisingly well for the stress tensor at the Schwarzschild horizon produces significant error in the noise kernel evaluated there. We check this using the trace anomaly expression and identify the failure as occurring at the fourth covariant derivative order.

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I. INTRODUCTION

The noise kernel is the vacuum expectation value of the stress-energy bitensor for a quantum field. In curved spacetime field theory [1–5] it plays a role in stochastic semiclassical gravity [6–13] similar to the expectation value of the stress-energy tensor in semiclassical gravity [14]. The noise kernel, being a measure of the fluctuations of the stress tensor of quantum fields, enters in a variety of issues and problems ranging from the validity of semiclassical gravity to spacetime foams, from structure formation in the early universe to fluctuations of the black hole horizon and trans-Planckian physics. The noise kernel in hot flat space, one of the two examples considered here, is useful for performing a real time quantum field theoretical analysis of the black hole nucleation problem (a preliminary result is reported in Ref. [15]) beyond that conducted in Euclidean-time thermal field dynamics [16,17]. In the other example of the Schwarzschild spacetime, knowledge of the noise kernel is essential for studying issues related to black hole horizon fluctuations [18–22] and Hawking radiation back reaction (see e.g., Ref. [23–26], and references therein).

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In paper I [27], we derived a general expression for the noise kernel of a quantum scalar field in an arbitrary curved spacetime as products of covariant derivatives of the quantum field's Green function. It is finite when the noise kernel is evaluated for distinct pairs of points (and non-null points for a massless field). We also showed explicitly that the trace of the noise kernel vanishes, confirming there is no noise associated with the trace anomaly. However, the noise kernel as a two point function of the stress-energy tensor diverges as the pair of points are brought together. This paper discusses how this divergence of the noise kernel can be dealt with for specific spacetimes of physical interest.

Before we embark on the details, it might be useful to recall why examining such a divergent quantity at the coincident limit is of interest. It is the long standing opinion of one of us (see, e.g., Ref. [9]) that field quantities defined at two separated points may describe the statistical mechanical properties of quantum fields [28] in a spacetime with possible extended structure. In the stochastic gravity program, the point-separated expression of the stress energy bitensor have fundamental physical meaning since it contains information on the fluctuations and correlation of quantum fields, and by consistency with the gravity sector, retains some coherent properties of quantum spacetimes when probed at a finer resolution or higher energies [29]. Taking this view, we may also gain a new perspective on ordinary quantum field theory defined on single points: The coincidence limit depicts the low energy limit of the full quantum theory of matter and spacetimes. Ordinary (pointwise) quantum field theory, as a part of semiclassical gravity, is the lowest level of approximation of a more complete theory of the microstructure of spacetime and matter. As such, it should be viewed not as fundamental, but as an effective theory. Even the way in which the conventional point-defined field theory emerges from the full theory when the two points (e.g., x and y in the noise kernel) are brought together is interesting. Here the nonlocal structure of spacetime and its impact on quantum field theory become the central issues. It is from this perspective that we arrive at a new understanding of the meaning of regularization and the coincident limit. This was discussed in paper I [27].

Apart from this philosophical bias, of practical interest is the quantity obtained from taking the coincident limit of the noise kernel, i.e., the fluctuations of energy density in ordinary (pointwise) quantum field theory—how does one define a finite quantity and make sense of it when addressing fundamental issues such as the validity of semiclassical gravity [30], e.g., whether the fluctuations to mean ratio is a correct criterion [31–35]. This is relevant to a study of the physical effects of black hole horizon fluctuations and Hawking radiation back reaction. It could also be important when determining whether certain inflationary models based on the fluctuations in the vacuum energy density can indeed be implemented.

Similar to what was done before for the simpler stress-energy tensor, to calculate a regularized noise kernel, it is desirable to have at the start an analytic and finite regularized expression for the Green function. When one can carry out a mode decomposition of the invariant operator and find an analytic solution to the mode functions there are established ways to proceed. For such cases the quantum stress tensor and its fluctuations can be determined using some regularization scheme, from the simple normal ordering [30,36] or smeared field [35] in Minkowski and Casimir spaces to the elegant ζ function or the powerful point-separation methods, as demonstrated for the Einstein Universe or for the Casimir effect by the authors [35,37] and others [38–40]. Unfortunately, there are many important geometries for which an exact analytic expression of the mode functions is not available, such as the Schwarzschild black hole spacetime.

We work with the covariant point-separation regularization method [41–43] in its modified form [44,45] to derive a finite expression for the coincident limit of the noise kernel. The expression derived in paper I [27] for the noise kernel of a scalar field is completely general and can be used with or without consideration of the regularization of the Green function. Also, the result there for the coincident limit holds for all choices of the Green function and the metric provided that the Green function has a meaningful coincident limit. In this paper, we apply this formal procedure to specific spacetimes of physical interest. We do this by working with an analytic form of the Green function. When such a form is available one can carry out an end point expansion displaying the ultraviolet divergence. Subtraction of the Hadamard ansatz expressed as a series expansion will render this Green function finite in the coincident limit. With this, one can calculate the noise kernel for a variety of spacetimes.

An analytic form is obtained by invoking the Gaussian approximation introduced by Bekenstein and Parker [46]. For a massless scalar field in an ultrastatic spacetime whose metric has an optical form (one where the Euclidean-time τ-time component of the metric $g_{\tau\tau}=1$) this provides a closed expression for the Green function. In this paper we use the Gaussian approximation for the Green function for such quantum fields to evaluate the noise kernel in two optical metrics: (1) hot flat space, (2) the optical-Schwarzschild spacetime, and then in (3) the Schwarzschild metric. For hot flat space, the Gaussian Green function is exact. For optical Schwarzschild, the Gaussian Green function is known to be a fairly good approximation for calculating the stress tensor [47], which involves second covariant derivatives of the Green function. We will carry out this calculation for the noise kernel which requires up to four covariant derivatives of the Green function. Thus the validity of the Gaussian approximation will be tested to its new limit. A reliable check is provided by the trace of the noise kernel, which for massless conformal fields should be zero.

We present an outline of the calculation as follows: After a brief description of the Gaussian approximation to the Green function [46] for ultrastatic spacetimes [47] in Sec. II we consider in Sec. III the regularization of the heat kernel in the class of optical metrics. We expand this Green function in an end point series so that it can be separated into a divergent piece and a finite remainder. The divergent term is independent of the approximation since we know this structure must be of a general form given by the Hadamard ansatz. After this is subtracted, we have a series expansion of the renormalized Green function. We then substitute this expansion in the general expression obtained in paper I for the coincident limit of the noise kernel. The resulting expression is quite long and formal. At this point one can introduce the specific metric of interest and determine the component values of the Green function expansion tensors (by symbolic computation). From this we can readily generate all the needed component values of the coincident limits of the covariant derivatives of the Green function, along with the covariant derivatives of the coincident limits. These explicitly evaluated tensors are then substituted in the general expression obtained in paper I to get the final result.[1]

We give two examples of the noise kernel in optical spacetimes in Sec. IV. For the case of hot flat space we derive the variance of the energy and pressure density for a quantum field at finite temperature. This is a useful compendium to the results obtained earlier [35] for quantum fields in Minkowski and Casimir geometries in reference to issues such as the validity of semiclassical gravity [30,31,34,35]. For a massless, conformally coupled field in the optical-

[1]We must stress, as we did in paper I, that though all these are done on a computer, no numerical approximation is used. All work is done symbolically in terms of the explicit functional form of the metric and the parameters of the field. The final results are exact to the extent that the analytic form of the Green function is exact.

Schwarzschild metric (the ultrastatic spacetime conformal to the Schwarzschild black hole), we obtain for the regularized noise kernel at spatial infinity the same result as a thermal field in flat space, as we should, and a finite result at the horizon in a state conformally related to the Hartle-Hawking state. However, the latter expression computed with the Gaussian approximation has a nonvanishing trace. In Sec. V we study the nature and source of this error by examining the validity of the Gaussian approximation at successive orders. It works reasonably well to the third covariant derivative order. The inadequacy of the Gaussian approximation to the Green function for the calculation of the noise kernel arises from the Green function's failure to satisfy the field equation of the scalar field at the fourth order.

In the second part of this paper (Secs. VI and VII) we treat spacetimes with metrics conformally related to an optical metric, focusing on the Schwarzschild black hole spacetime. For a given noise kernel in an optical metric, one can take advantage of the simple conformal transformation property of the scalar field's Green function to compute the noise kernel for the corresponding conformally optical metrics. According to the procedure described above, the main obstacle is the subtraction of the Hadamard ansatz. The divergent Green function is defined in terms of the optical metric while the Hadamard ansatz is defined in terms of the physical metric. Thus we need to reexpress the transformed optical metric in terms of the physical metric. The defining equations for the geometric objects on the optical metric are transformed to the physical metric and solved recursively. Now the Green function series expansion can be written solely in terms of the physical metric. While for the ultrastatic spacetimes the Hadamard subtraction was straightforward, for spacetimes only conformally ultrastatic this subtraction is nontrivial. Before the subtraction can be carried out and the Green function regularized, we need to study the conformal transformation properties of the point-separation objects used to define the Green function.[2]

Since all the series expansions used are recursively derived, as a check of the full expression it is sufficient that we get the correct results for the lower order terms. Indeed our general expression contains and confirms the Page [47] result for the vacuum expectation value of the stress tensor in the Schwarzschild black hole. With the general expansion of the renormalized Green function at hand, one can choose a metric and compute the coincident limit of the noise kernel as outlined above. In Sec. VI we compute the noise kernel for spacetimes conformally related to ultrastatic spacetimes. In Sec. VII we specialize regarding the Schwarzschild black hole and discuss the significance of our results. The Appendices contain expressions useful for our considerations here. More details can be found in Ref. [48].

[2]As will be shown, regularizing the divergent structure to sufficient order for the noise kernel is no small task, when we include the conformal transformation between the physical and the optical metrics (e.g., the fourth order term in the expansion of the renormalized Green function has over 1100 terms). It is at this point that the symbolic computation environment takes over.

II. GAUSSIAN APPROXIMATION

We give a brief description of the Gaussian approximation to the Green function for quantum fields in optical spacetimes in the manner of Bekenstein-Parker [46] and Page [47].

In the Schwinger-DeWitt proper-time formalism [41,49] the Green function is expressed in terms of the heat kernel $K(x,y,s)$ via

$$G(x,y)=\int_0^\infty K(x,y,s)\,ds, \tag{2.1}$$

where the heat kernel satisfies

$$\left[\frac{\partial}{\partial s}-\left(\Box-\frac{R}{6}\right)\right]K(x,y,s)=0, \quad K(x,y,0)=\delta(x-y). \tag{2.2}$$

The optical metric for an ultrastatic spacetime has the product form

$$ds^2=g_{ab}dx^a dx^b=d\tau^2+g_{ij}dx^i dx^j. \tag{2.3}$$

We assume in the Euclidean sector that the imaginary time dimension is periodic with period $2\pi/\kappa=1/T$, with T the temperature. For a black hole, κ is the surface gravity but it can be regarded as a temperature parameter here. This form of the metric allows the kernel to take on the product form

$$K(x,y,s)=K_1(\tau,\tau',s)K_3(\mathbf{x},\mathbf{y},s) \tag{2.4}$$

with each of the kernels satisfying

$$\left(\frac{\partial}{\partial s}-\frac{\partial^2}{\partial\tau^2}\right)K_1(\tau,\tau',s)=0, \tag{2.5a}$$

$$\left[\frac{\partial}{\partial s}-\left(\nabla_i\nabla^i-\frac{R}{6}\right)\right]K_3(\mathbf{x},\mathbf{y},s)=0. \tag{2.5b}$$

Equation (2.5a) has the periodic solution

$$K_1(\tau,\tau',s)=\frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty}\exp(-\kappa^2 n^2 s+i\kappa n\Delta\tau) \tag{2.6}$$

($\Delta\tau=\tau-\tau'$). Equation (2.5b) in general is difficult to solve, but Bekenstein and Parker [46] find an approximate solution using the Gaussian approximation to the path integral representation. For K_3, it takes the form

$$K_{3\mathrm{Gauss}}(\mathbf{x},\mathbf{y},s)=\frac{{}^{(3)}\Delta^{1/2}}{(4\pi s)^{3/2}}\exp\left[-\frac{{}^{(3)}\sigma}{2s}\right] \tag{2.7}$$

where ${}^{(3)}\sigma$ is the world function for the three-dimensional spatial geometry. In the above, ${}^{(3)}\Delta^{1/2}$ is the Van Vleck–Morette determinant for the spatial geometry. Since we have an optical metric, the four-dimensional world function is $\sigma={}^{(3)}\sigma+\Delta\tau^2/2$ and there is no difference between the three and four dimensional $\Delta^{1/2}$.

Since the complete Hadamard-Minakshisundaram-Pleijel-DeWitt expansion [41] would be

$$K_3 = K_{3\text{Gauss}} \sum_{n=0}^{\infty} a_n(\mathbf{x},\mathbf{y}) s^n \tag{2.8}$$

the Gaussian approximation is equivalent to only taking the first term in this power series.

By putting Eqs. (2.6) and (2.7) back into Eq. (2.4) and carrying out the integration (2.1), Page obtains

$$G_{\text{Gauss}}(x,y) = \frac{\kappa \Delta^{1/2}}{8\pi^2 r} \frac{\sinh \kappa r}{(\cosh \kappa r - \cos \kappa \tau)} \tag{2.9}$$

as the Gaussian approximation for the Green function, where $r = [2\,{}^{(3)}\sigma]^{1/2}$.

III. NOISE KERNEL IN OPTICAL SPACETIMES

In this section, the noise kernel for an ultrastatic spacetime with an optical metric is determined. For this class of geometries, we start directly with Eq. (2.9) for the Green function. The first step is to expand this Green function about the coincident limit. Since the noise kernel has terms with at most four covariant derivatives, this expansion needs to be to fourth order in σ^a, or second order in $\sigma = \sigma^p \sigma_p / 2$, and fourth order in $\Delta\tau$. Doing this expansion yields

$$G_{\text{Gauss}} = \frac{\Delta^{1/2}}{8\pi^2 \sigma} + \frac{\Delta^{1/2}}{8\ \pi^2} \left\{ \frac{\kappa^2}{6} + \frac{\kappa^4}{180}(2\Delta\tau^2 - \sigma) + \frac{\kappa^6}{3780}(4\Delta\tau^4 - 6\Delta\tau^2\sigma + \sigma^2) \right\} + O(\sigma^{5/2}, \delta\tau^5). \tag{3.1}$$

By subtracting from the Gaussian Green function the Hadamard ansatz

$$S(x,y) = \frac{1}{16\pi^2}\left(\frac{2\Delta^{1/2}}{\sigma} + \sigma w_1 + \sigma^2 w_2 \right) + O(\sigma^3) \tag{3.2}$$

[the $V(x,x')$ term is absent since there is no log σ divergence present in the expansion of the Gaussian approximation to the Green function] we get the renormalized Green function

$$G_{\text{ren}} = G_{\text{Gauss}} - S. \tag{3.3}$$

The divergent term present (3.1) is canceled by the divergent term from the Hadamard ansatz.

Since our main interest here is to determine the coincident limit of the noise kernel, we next turn to developing the series expansion

$$G_{\text{ren}} = \frac{1}{(4\pi)^2}\Big(G^{(0)}_{\text{ren}} + \sigma^{;p} G^{(1)}_{\text{ren}p} + \sigma^{;p}\sigma^{;q} G^{(2)}_{\text{ren}pq} + \sigma^{;p}\sigma^{;q}\sigma^{;r} G^{(3)}_{\text{ren}pqr} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s} G^{(4)}_{\text{ren}pqrs} \Big) \tag{3.4}$$

of the regularized Green function. With this, it will be straightforward to compute the coincident limits of the various covariant derivatives needed. We start by assuming the expansions

$$\Delta^{1/2} \approx 1 + \sigma^{;p}\sigma^{;q}\Delta^{(2)}{}_{pq} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\Delta^{(3)}{}_{pqr} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}\Delta^{(4)}{}_{pqrs}, \tag{3.5a}$$

$$\Delta\tau^2 \approx \sigma^{;p}\sigma^{;q}\delta\tau^{(2)}{}_{pq} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\delta\tau^{(3)}{}_{pqr} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}\delta\tau^{(4)}{}_{pqrs}, \tag{3.5b}$$

$$w_1 \approx w_1^{(0)} + \sigma^{;p} w_1^{(1)}{}_p + \sigma^{;p}\sigma^{;q} w_1^{(2)}{}_{pq}, \tag{3.5c}$$

$$w_2 \approx w_2^{(0)}. \tag{3.5d}$$

The specific values of the expansion tensors in these series are derived in the Appendices. Carrying out the subtraction (3.3) and substituting the expansions (3.5), we find that the expansions tensors in Eq. (3.4) are

$$G^{(0)}_{\text{ren}} = \frac{\kappa^2}{3}, \tag{3.6a}$$

$$G^{(1)}_{\text{ren}a} = 0, \tag{3.6b}$$

$$G^{(2)}_{\text{ren}ab} = \frac{\kappa^2}{3}\Delta^{(2)}{}_{ab} + \frac{\kappa^4}{180}[4\delta\tau^{(2)}{}_{ab} - g_{ab}] - \frac{1}{2} w_1^{(0)} g_{ab}, \tag{3.6c}$$

$$G^{(3)}_{\text{ren}abc} = \frac{\kappa^2}{3}\Delta^{(3)}{}_{abc} + \frac{\kappa^4}{45}\delta\tau^{(3)}{}_{abc} - \frac{1}{2} g_{ab} w_1^{(1)}{}_c, \tag{3.6d}$$

$$G^{(4)}_{\text{ren}abcd} = \frac{\kappa^2}{3}\Delta^{(4)}{}_{abcd} + \frac{\kappa^4}{180}[4\Delta^{(2)}{}_{ab}\delta\tau^{(2)}{}_{cd} + 4\delta\tau^{(4)}{}_{abcd} - \Delta^{(2)}{}_{cd} g_{ab}] + \frac{\kappa^6}{7560}[16\delta\tau^{(2)}{}_{ab}\delta\tau^{(2)}{}_{cd} - 12\delta\tau^{(2)}{}_{cd} g_{ab} + g_{ab} g_{cd}] - \frac{1}{4} g_{ab}[w_2^{(0)} g_{cd} + 2 w_1^{(2)}{}_{cd}]. \tag{3.6e}$$

Using the explicit forms of the expansion tensor values, Eqs. (C8), (C10), (C12a), and (D5), we get

$$G^{(2)}_{\text{ren}ab} = \frac{\kappa^2}{36} R_{ab} + \frac{\kappa^4}{180}(4\delta_a{}^\tau \delta_b{}^\tau - g_{ab}), \tag{3.7a}$$

$$G^{(3)}_{\text{ren}abc} = -\frac{\kappa^2}{72} R_{ab;c} + \frac{\kappa^4}{45}\Gamma^\tau{}_{ab}\delta_c{}^\tau), \tag{3.7b}$$

$$G^{(4)}_{\mathrm{ren}abcd}=\frac{\kappa^2}{4320}(18R_{ab;cd}+5R_{ab}R_{cd}+4R_{paqb}R_c{}^p{}_d{}^q)$$
$$+\frac{\kappa^4}{2160}(12\Gamma^\tau{}_{ab}\Gamma^\tau{}_{cd}-16\Gamma^\tau{}_{ab;c}\delta_d{}^\tau$$
$$+4\delta_a{}^\tau\delta_b{}^\tau R_{cd}-g_{ab}R_{cd})$$
$$+\frac{\kappa^6}{7560}(16\delta_a{}^\tau\delta_b{}^\tau\delta_c{}^\tau\delta_d{}^\tau-12\delta_a{}^\tau\delta_b{}^\tau g_{cd}$$
$$+g_{ab}g_{cd})-\frac{1}{4}g_{ab}(w_2^{(0)}g_{cd}+2w_1^{(2)}{}_{cd}), \qquad (3.7c)$$

where we have also used

$$w_1^{(0)}=-\frac{1}{240}(R_{;p}{}^p-R_{pq}R^{pq}+R_{pqrs}R^{pqrs})=0, \qquad (3.8)$$

along with $w_1^{(1)}{}_a=0$, which hold for ultrastatic metrics.

For flat space, these tensors reduce to

$$G^{(2)}_{\mathrm{ren}ab}=\frac{\kappa^4}{180}(-\eta_{ab}+4\eta_a{}^\tau\eta_b{}^\tau), \qquad (3.9a)$$

$$G^{(3)}_{\mathrm{ren}abc}=0, \qquad (3.9b)$$

$$G^{(4)}_{\mathrm{ren}abcd}=\frac{\kappa^6}{7560}(\eta_{ab}\eta_{cd}-12\eta_a{}^\tau\eta_b{}^\tau\eta_{cd}$$
$$+16\eta_a{}^\tau\eta_b{}^\tau\eta_c{}^\tau\eta_d{}^\tau]. \qquad (3.9c)$$

Now that we know the end point series expansion (3.4) of G_{ren}, the coincident limit of terms with up to four covariant derivatives are computed. We simply differentiate the series (3.4) and then use the results from Appendix A for the coincident limits of the covariant derivatives of the world function σ. The results are

$$16\pi^2[G_{\mathrm{ren}}]=G^{(0)}_{\mathrm{ren}}, \qquad (3.10a)$$

$$16\pi^2[G_{\mathrm{ren};a}]=G^{(0)}_{\mathrm{ren};a}, \qquad (3.10b)$$

$$16\pi^2[G_{\mathrm{ren};ab}]=G^{(0)}_{\mathrm{ren}(ab)}+2G^{(2)}_{\mathrm{ren}(ab)}, \qquad (3.10c)$$

$$16\pi^2[G_{\mathrm{ren};abc}]=6(G^{(2)}_{\mathrm{ren}(ab;c)}+G^{(3)}_{\mathrm{ren}(abc)})+G^{(0)}_{\mathrm{ren};abc}, \qquad (3.10d)$$

$$16\pi^2[G_{\mathrm{ren};abcd}]=12(2G^{(3)}_{\mathrm{ren}(abc;d)}+G^{(2)}_{\mathrm{ren}(ab;cd)}$$
$$+2G^{(4)}_{\mathrm{ren}(abcd)})+\frac{2}{3}(G^{(2)}_{\mathrm{ren}pa}(R_{bcd}{}^p$$
$$-2R_{bdc}{}^p)+G^{(2)}_{\mathrm{ren}pb}(R_{acd}{}^p-2R_{adc}{}^p)$$
$$+G^{(2)}_{\mathrm{ren}pc}(R_{abd}{}^p-2R_{adb}{}^p)+G^{(2)}_{\mathrm{ren}pd}(R_{abc}{}^p$$
$$-2R_{acb}{}^p))+G^{(0)}_{\mathrm{ren};abcd}. \qquad (3.10e)$$

We now have all the information we need to compute the coincident limit of the noise kernel [see Eq. (3.24) of paper I]. Since the point-separated noise kernel $N_{abc'd'}(x,y)$ involves covariant derivatives at the two points at which it has support, when we take the coincident limit we can use Synge's theorem to move the derivatives acting at the second point y to ones acting at the first point x. Because of the length of the expression for the noise kernel, we will here give an example of the calculation by examining a single term. The complete expression for the coincident limit of the point-separated noise kernel can be found in paper I as Eq. (4.16).

Consider a typical term from the noise kernel functional [Eq. (3.24) of paper I]:

$$G_{\mathrm{ren};c'b}G_{\mathrm{ren};d'a}+G_{\mathrm{ren};c'a}G_{\mathrm{ren};d'b}. \qquad (3.11)$$

As was derived in paper I, the noise kernel itself is related to the noise kernel functional via

$$N_{abc'd'}=N_{abc'd'}[G_{ren}(x,y)]+N_{abc'd'}[G_{ren}(y,x)]. \qquad (3.12)$$

We account for this by adding the same term but with the roles of x and y reversed. Taken together, we need to analyze

$$G_{\mathrm{ren};c'b}G_{\mathrm{ren};d'a}+G_{\mathrm{ren};a'd}G_{\mathrm{ren};b'c}+G_{\mathrm{ren};c'a}G_{\mathrm{ren};d'b}$$
$$+G_{\mathrm{ren};a'c}G_{\mathrm{ren};b'd}, \qquad (3.13)$$

in particular, its coincident limit:

$$[G_{\mathrm{ren};c'b}][G_{\mathrm{ren};d'a}]+[G_{\mathrm{ren};a'd}][G_{\mathrm{ren};b'c}]$$
$$+[G_{\mathrm{ren};c'a}][G_{\mathrm{ren};d'b}]+[G_{\mathrm{ren};a'c}][G_{\mathrm{ren};b'd}]. \qquad (3.14)$$

We apply Synge's theorem to remove any explicit reference to point y,

$$([G_{\mathrm{ren};a}]_{;d}-[G_{\mathrm{ren};ad}])([G_{\mathrm{ren};b}]_{;c}-[G_{\mathrm{ren};bc}])$$
$$+([G_{\mathrm{ren};d}]_{;a}-[G_{\mathrm{ren};ad}])([G_{\mathrm{ren};c}]_{;b}-[G_{\mathrm{ren};bc}])$$
$$+([G_{\mathrm{ren};a}]_{;c}-[G_{\mathrm{ren};ac}])([G_{\mathrm{ren};b}]_{;d}-[G_{\mathrm{ren};bd}])$$
$$+([G_{\mathrm{ren};c}]_{;a}-[G_{\mathrm{ren};ac}])([G_{\mathrm{ren};d}]_{;b}-[G_{\mathrm{ren};bd}]), \qquad (3.15)$$

express the results in terms of the expansion tensors (3.10e) and find

$$8(G^{(2)}_{\mathrm{ren}ad}G^{(2)}_{\mathrm{ren}bc}+G^{(2)}_{\mathrm{ren}ac}G^{(2)}_{\mathrm{ren}bd}). \qquad (3.16)$$

Though this may look relatively compact, when we use the results (3.7c) to get the final form, in terms of the local geometry, this is no longer the case. Making these substitutions, our pair of terms becomes

$$\frac{\kappa^4}{2592\pi^2}(R_{ad}R_{bc}+R_{ac}R_{bd})+\frac{\kappa^6}{12960\pi^2}\{(4\delta_b{}^\tau\delta_d{}^\tau-g_{bd})R_{ac}+(4\delta_b{}^\tau\delta_c{}^\tau-g_{bc})R_{ad}+(4\delta_a{}^\tau\delta_d{}^\tau-g_{ad})R_{bc}+(4\delta_a{}^\tau\delta_c{}^\tau-g_{ac})R_{bd}\}+\frac{\kappa^8}{64800\pi^2}\{32\delta_a{}^\tau\delta_b{}^\tau\delta_c{}^\tau\delta_d{}^\tau+g_{ad}g_{bc}+g_{ac}g_{bd}-4(\delta_b{}^\tau\delta_d{}^\tau g_{ac}+\delta_b{}^\tau\delta_c{}^\tau g_{ad}+\delta_a{}^\tau\delta_d{}^\tau g_{bc}+\delta_a{}^\tau\delta_c{}^\tau g_{bd})\}. \tag{3.17}$$

The noise kernel functional consists of 25 such terms, some quite a bit more involved. This is especially true when a single Green function has four derivatives acting on it. To gain insight into the physics we work with some specific spacetime.[3]

When we do choose a metric, we proceed by directly evaluating the components of the expansion tensors (3.7c). Once these expansion tensor components are known, the actual components of the coincident limits of the covariant derivatives of $G_{\rm ren}$ are in turn computed using (3.10e). Then it is straightforward from there to get the covariant derivatives of the coincident limits, since application of Synge's theorem will move derivatives acting at y such that we also need the covariant derivatives of the coincident limits. Now that we have the component values of all the covariant derivatives of the various coincident limits of the regularized Green function $G_{\rm ren}$, we substitute them in the coincident limit expression for the noise kernel, Eq. (4.16) of paper I, and arrive at the final result we seek.

Before turning to specific metrics and as a check, we can reproduce the derivation of the renormalized stress tensor. We start with the point-separated expression for the stress tensor, which for a massless, conformally coupled scalar field is

$$\langle T_{ab}(x,y)\rangle=\frac{1}{3}(g^{p'}{}_bG_{{\rm ren};p'a}+g^{p'}{}_aG_{{\rm ren};p'b})-\frac{1}{6}g^{p'}{}_qG_{{\rm ren};p'}{}^q g_{ab}-\frac{1}{6}(g^{p'}{}_ag^{q'}{}_bG_{{\rm ren};p'q'}+G_{{\rm ren};ab})+\frac{1}{6}((G_{{\rm ren};p'}{}^{p'}+G_{{\rm ren};p}{}^p)g_{ab})+\frac{1}{6}G_{\rm ren}\left(\frac{-(Rg_{ab})}{2}+R_{ab}\right). \tag{3.18}$$

We take the coincident limit and utilize Synge's theorem to obtain

$$\langle T_{ab}\rangle_{\rm ren}=\frac{1}{6}(3[G_{{\rm ren};a}]_{;b}+3[G_{{\rm ren};b}]_{;a}-[G_{\rm ren}]_{;ab}-6[G_{{\rm ren};ab}])-\frac{1}{6}(3[G_{{\rm ren};p}]^{;p}-[G_{\rm ren}]_{;p}{}^p-3[G_{{\rm ren};p}{}^p])g_{ab}-\frac{1}{12}[G_{\rm ren}](Rg_{ab}-2R_{ab}),$$
$$=-\frac{1}{6}(G^{(0)}_{{\rm ren};ab}+12G^{(2)}_{{\rm ren}ab}-G^{(0)}_{\rm ren}R_{ab})-\frac{1}{12}(G^{(0)}_{\rm ren}R-2G^{(0)}_{{\rm ren};p}{}^p-12G^{(2)}_{{\rm ren}p}{}^p)g_{ab}. \tag{3.19}$$

With the explicit values (3.7c) for the expansion tensors we recover the familiar result

$$\langle T_{ab}\rangle_{\rm ren}=\frac{\kappa^4}{1440\pi^2}(g_{ab}-4\delta_a{}^\tau\delta_b{}^\tau). \tag{3.20}$$

IV. EXAMPLES

A. Hot flat space

The first example we consider is that of a finite temperature $T=\kappa/2\pi$ quantum scalar field in flat space. With this, the stress tensor takes the usual form

$$\langle T_{ab}\rangle=\mathrm{diag}\left\{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},1\right\}\rho,\qquad \rho=-\frac{\pi^2T^4}{30}, \tag{4.1}$$

with $x^a=(x,y,z,\tau)$.

Using Eqs. (3.9c) and (3.10e), the nonzero coincident limits of the derivatives of $G_{\rm ren}$ are

$$[G_{\rm ren}]=\frac{T^2}{12}, \tag{4.2a}$$

$$[G_{{\rm ren};ab}]=-\frac{(\pi^2T^4)}{90}(\delta_{ab}-4\delta_a{}^\tau\delta_b{}^\tau), \tag{4.2b}$$

$$[G_{{\rm ren};abcd}]=\frac{4\pi^4T^6}{315}(\delta_{ab}\delta_{cd}-12\delta_a{}^\tau\delta_b{}^\tau\delta_{cd}+16\delta_a{}^\tau\delta_b{}^\tau\delta_c{}^\tau\delta_d{}^\tau). \tag{4.2c}$$

Taking the coincident limit of the massless case for the noise kernel, Eq. (4.16) of paper I (for this case, we keep ξ arbitrary), we find

[3]Since this work for analyzing the coincident limit of the noise kernel via the point-separation method is tailored to symbolic computation on the computer, our method for deriving particular results is designed to take maximum advantage of the computer.

$$N_{abcd}=\frac{\pi^4T^8}{226800}\{32(7-28\xi+162\xi^2)g^\tau{}_a g^\tau{}_b g^\tau{}_c g^\tau{}_d-8(7-42\xi+130\xi^2)(g_{cd}g^\tau{}_a g^\tau{}_b+g_{ab}g^\tau{}_c g^\tau{}_d)-4(7-28\xi+148\xi^2)(g_{bd}g^\tau{}_a g^\tau{}_c+g_{ad}g^\tau{}_b g^\tau{}_c+g_{bc}g^\tau{}_a g^\tau{}_d+g_{ac}g^\tau{}_b g^\tau{}_d)+4(7-63\xi+167\xi^2)g_{ab}g_{cd}+(7-28\xi+108\xi^2)(g_{ad}g_{bc}+g_{ac}g_{bd})\}. \tag{4.3}$$

From this, we can immediately compute the trace:

$$N_p{}^p{}_q{}^q=\frac{\pi^4T^8}{1350}(-1+6\xi)^2, \tag{4.4}$$

which we see vanishes for the conformal coupling case ($\xi=1/6$). The nonvanishing components for general coupling are

$$N_{\tau\tau\tau\tau}=\frac{\pi^4T^8}{37800}(7-14\xi+270\xi^2)\overset{\xi\to0}{\to}\frac{\pi^4T^8}{5400}=\frac{1}{6}\langle T_{\tau\tau}\rangle^2,$$

$$N_{xxxx}=\frac{\pi^4T^8}{113400}(21-154\xi+442\xi^2)\overset{\xi\to0}{\to}\frac{\pi^4T^8}{5400}=\frac{3}{2}\langle T_{xx}\rangle^2,$$

$$N_{\tau\tau xx}=-\frac{\pi^4T^8}{56700}(7-21\xi+93\xi^2)\overset{\xi\to0}{\to}-\frac{\pi^4T^8}{8100}=\frac{1}{3}\langle T_{\tau\tau}\rangle\langle T_{xx}\rangle,$$

$$N_{xxyy}=\frac{\pi^4T^8}{56700}(7-63\xi+167\xi^2)\overset{\xi\to0}{\to}\frac{\pi^4T^8}{8100}=1\langle T_{xx}\rangle^2,$$

$$N_{\tau x\tau x}=-\frac{\pi^4T^8}{226800}(21-84\xi+484\xi^2)\overset{\xi\to0}{\to}-\frac{\pi^4T^8}{10800}=\frac{1}{4}\langle T_{\tau\tau}\rangle\langle T_{xx}\rangle,$$

$$N_{xyxy}=\frac{\pi^4T^8}{226800}(7-28\xi+108\xi^2)\overset{\xi\to0}{\to}\frac{\pi^4T^8}{32400}=\frac{1}{4}\langle T_{xx}\rangle^2, \tag{4.5}$$

and those that follow from the symmetry of the metric.

We use the dimensionless measure of fluctuations [30,35–37] (this is not a tensor, but a measure of the fluctuations for each component):

$$\Delta_{abcd}=\left|\frac{\langle T_{ab}T_{cd}\rangle-\langle T_{ab}\rangle\langle T_{cd}\rangle}{\langle T_{ab}T_{cd}\rangle}\right|=\left|\frac{4N_{abcd}}{4N_{abcd}+\langle T_{ab}\rangle\langle T_{cd}\rangle}\right|. \tag{4.6}$$

From inspection, $0\leqslant\Delta_{abcd}\leqslant1$. Only for $\Delta\sim0$ can the fluctuations be viewed as small. For $\Delta\sim1$ the fluctuations are comparable to the mean value.

For hot flat space, the results for Δ are

$abcd$:	Δ_{abcd},	$\xi=0$,	$\xi=\frac{1}{6}$,
$\tau\tau\tau\tau$:	$\frac{2(7-14\xi+270\xi^2)}{35-28\xi+540\xi^2}$,	$\frac{2}{5}\sim0.4$,	$\frac{73}{136}\sim0.54$,
$xxxx$:	$\frac{2(21-154\xi+442\xi^2)}{49-308\xi+884\xi^2}$,	$\frac{6}{7}\sim0.86$,	$\frac{137}{200}\sim0.69$,
$\tau\tau xx$:	$\frac{4(7-21\xi+93\xi^2)}{49-84\xi+372\xi^2}$,	$\frac{4}{7}\sim0.57$,	$\frac{73}{136}\sim0.54$,
$xxyy$:	$\frac{4(7-63\xi+167\xi^2)}{35-252\xi+668\xi^2}$,	$\frac{4}{5}\sim0.8$,	$\frac{41}{104}\sim0.39$.

(4.7)

From these results we see that, even for the simple case of thermal fluctuations in flat space, the fluctuations present in the stress tensor are important. Discussions on the implication of the fluctuation to mean ratio can be found in Refs. [30,31,34–36].

B. Optical-Schwarzschild black hole

We now consider the optical spacetime conformally related to the Schwarzschild black hole spacetime. For this spacetime, the line element is

$$ds^2=d\tau^2+\left(1-\frac{2M}{r}\right)^{-2}dr^2+\left(1-\frac{2M}{r}\right)^{-1}r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{4.8}$$

Taking $\kappa=2\pi T$ and $T=1/(8\pi M)$ we choose the quantum state corresponding to the Hartle-Hawking state in the conformally related Schwarzschild spacetime. We use the spacetime coordinates $x^a=(r,\theta,\phi,\tau)$ and introduce the rescaled inverse radial coordinate $x=2M/r=1/(4\pi Tr)$. Spatial infinity corresponds to $x=0$, and $x=1$ is the black hole horizon. For a massless conformal scalar field ($m=0,\xi=1/6$) the stress tensor is

$$\langle T^b_a\rangle=\mathrm{diag}\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3},-1\right\}\frac{\pi^2T^4}{30}. \tag{4.9}$$

We recover the standard thermal result for the stress tensor. The component values for the noise kernel are

$$N_\tau{}^\tau{}_\tau{}^\tau=\frac{\pi^4T^8}{2041200}(657-1050x^4+8400x^6-16800x^7-16997400x^8+80206000x^9-140910000x^{10}+109242000x^{11}-31515750x^{12}), \tag{4.10a}$$

$$N_{r\ r}^{\ r\ r}=\frac{\pi^4T^8}{2041200}(137+1120x^3-210x^4+80640x^5-103600x^6+16800x^7+9829800x^8-43285200x^9+73602000x^{10}-57186000x^{11}+17057250x^{12}), \tag{4.10b}$$

$$N_{\theta\ \theta}^{\ \theta\ \theta}=\frac{\pi^4T^8}{2041200}(137-560x^3+1470x^4+30240x^5-44800x^6-16800x^7-15805800x^8+73203600x^9-127386000x^{10}+98820000x^{11}-28788750x^{12}), \tag{4.10c}$$

$$N_{\tau\ r}^{\ \tau\ r}=\frac{\pi^4T^8}{6123600}(-657+6720x^3-5670x^4-42000x^6+484920x^8+4202480x^9-12514800x^{10}+9439200x^{11}-1548450x^{12}), \tag{4.10d}$$

$$N_{\tau\ \theta}^{\ \tau\ \theta}=\frac{\pi^4T^8}{6123600}(-657-3360x^3+4410x^4+8400x^6+25200x^7-53478360x^8+248828960x^9-434589600x^{10}+337051800x^{11}-97825050x^{12}), \tag{4.10e}$$

$$N_{r\ \theta}^{\ r\ \theta}=\frac{\pi^4T^8}{6123600}(123-5040x^3+3150x^4-120960x^5+176400x^6-25200x^7+2858760x^8-10502560x^9+15885600x^{10}-12447000x^{11}+4178250x^{12}). \tag{4.10f}$$

From the component values of the noise kernel we can compute its trace,

$$N=N_{p\ q}^{\ p\ q}=-\frac{4\pi^4T^8x^8}{567}(9720-45832x+80520x^2-62424x^3+18009x^4). \tag{4.11}$$

We know from prior results that this should vanish, since we have worked with a massless conformally coupled scalar field. This failure of the trace of the noise kernel to vanish is due to the failure of the Gaussian approximated Green function (2.9) to satisfy the field equation to fourth order.

V. FAILURE OF GAUSSIAN APPROXIMATION AT THE FOURTH ORDER

To be sure that this error does not arise from the symbolic manipulation, let us mention ways to check the correctness of the algorithm. The basic procedures for generating the needed series expansions are recursive on the expansion order. (The recursion formulas of the expansion used in point separation are collected in the Appendices.) For the noise kernel we need results up to fourth order in the separation distance. The well established work for the stress tensor is to second order. This provides a check of our code by verifying we always get the known results for the stress tensor expectation value. Once we know that the second order recursion is correct, we know the algorithm is functioning as desired. (The correctness in the (new) fourth order terms becomes particularly important in the conformally optical metric calculations, since we get intermediate results of up to 1100 terms in length.)

We can check the accuracy of the Gaussian approximation by using the computed component values of the coincident limit of the covariant derivatives of $G_{\rm ren}$. We have assumed

$$G_{{\rm ren};p}{}^{p}-\frac{G_{\rm ren}R}{6}=0. \tag{5.1}$$

This can be used to test the approximation order by order. To test up to the second order, we just take the coincident limit

$$[G_{{\rm ren};p}{}^{p}]-\frac{R[G_{\rm ren}]}{6}=0. \tag{5.2}$$

For the metric (4.8), the scalar curvature is

$$R=-\frac{6M^2}{r^4}=-24\pi^2T^2x^4, \tag{5.3}$$

while the results of our computation of the component values of the coincident limit of the covariant derivatives of $G_{\rm ren}$ yield

$$[G_{\rm ren}]=\frac{4\pi^2T^2}{3},\quad [G_{{\rm ren};p}{}^{p}]=-\frac{16\pi^4T^4x^4}{3}. \tag{5.4}$$

With these values, Eq. (5.2) can be seen to be satisfied. Thus the Gaussian approximation is good to the second order. (This had better be the case, since the approximation at this order has been checked against numerical computations of the stress tensor by other authors. See, e.g., a description in Ref. [24].)

To check the third order term, we take one covariant derivative and then the coincident limit of Eq. (5.1):

$$[G_{{\rm ren};p}{}^{p}{}_{a}]-\frac{1}{6}(R_{;a}[G_{\rm ren}]+R[G_{{\rm ren};a}])=0. \tag{5.5}$$

Using the results $[G_{{\rm ren};a}]=0$,

$$R_{;a}=\{384\pi^3T^3x^5,0,0,0\}, \tag{5.6}$$

$$[G_{{\rm ren};p}{}^{p}{}_{a}]=\left\{\frac{256\pi^5T^5x^5}{3},0,0,0\right\}, \tag{5.7}$$

we see Eq. (5.5) is also satisfied. This has to be the case: Eq. (3.7c) shows that the third order expansion tensor does not have any contribution from the $W(x,y)$ part of the Hadamard ansatz and this is the only place a lack of symmetry in

$G_{\rm ren}(x,y)$ could appear. Therefore the Gaussian Green function (2.9) is symmetric. For symmetric functions, the odd order expansion tensors are determined completely by the even order tensors [see Eq. (B16a)].

Continuing to fourth order, Eq. (5.1) becomes

$$[G_{\mathrm{ren};p}{}^{p}{}_{a}{}^{b}]-\frac{1}{6}(R_{;a}{}^{b}[G_{\rm ren}]+R^{;b}[G_{\mathrm{ren};a}]+R_{;a}[G_{\rm ren}{}^{;b}]+R[G_{\mathrm{ren};a}{}^{b}])=[G_{\mathrm{ren};p}{}^{p}{}_{a}{}^{b}]-\frac{1}{6}(R_{;a}{}^{b}[G_{\rm ren}]+R[G_{\mathrm{ren};a}{}^{b}])=0. \tag{5.8}$$

Proceeding as before and evaluating the left hand side above, the component values are

$$\mathrm{diag}\Big\{-\frac{128\pi^6T^6x^8}{315}(162648-746888x+1295880x^2-1005480x^3+293805x^4),\frac{128\pi^6T^6x^8}{315}(8424-29704x+44040x^2-34560x^3+11835x^4),\frac{128\pi^6T^6x^8}{315}(8424-29704x+44040x^2-34560x^3+11835x^4),\frac{128\pi^6T^6x^8}{189}(9720-45832x+80520x^2-62424x^3+18009x^4). \tag{5.9}$$

The failure of the left hand side of Eq. (5.8) to vanish shows that the failure of the trace to vanish comes from the limitations of the Gaussian approximation. The Gaussian approximation is only useful up to the third order in σ^a.

With this knowledge, the trace (4.11) becomes our measure of the error in the noise kernel from the use of the Gaussian approximation. It is important to note that the noise kernel trace N vanishes as $x\to 0$, or, $r\to\infty$, i.e., where one would expect the effects of curvature to vanish. We can also see from Eq. (4.10f) that the noise is finite at the horizon $(x=1)$.

Using our derived expression for the noise kernel we see that its trace vanishes at spatial infinity, thus we can trust our results there. Using the measure (4.6), the fluctuations at $r\to\infty$ are

$$\begin{array}{lllllll} abcd: & \tau\tau\tau\tau & rrrr & \theta\theta\theta\theta & \tau\tau rr & \tau\tau\theta\theta & rr\theta\theta, \\ \Delta_{abcd}: & \frac{73}{136} & \frac{137}{200} & \frac{137}{200} & \frac{73}{136} & \frac{73}{136} & \frac{41}{104}, \end{array} \tag{5.10}$$

which match exactly the results (4.7) for hot flat space with conformal coupling, another reassuring fact. Since the computation of the noise kernel for the metric (4.8) is much more involved, this provides yet another check of our symbolic computer code.

Now having shown that it is truly the Gaussian approximation that is at fault for the failure of the noise kernel trace to vanish, we can use its value as a measure of the error in the results (4.10f). Since N should be zero, N/N_{abcd} is a dimensionless measure of this error. At the horizon $(x=1)$,

$$\begin{array}{lllllll} abcd: & \tau\tau\tau\tau & rrrr & \theta\theta\theta\theta & \tau\tau rr & \tau\tau\theta\theta & rr\theta\theta, \\ \dfrac{N_{p}{}^{p}{}_{q}{}^{q}}{N_{a}{}^{b}{}_{c}{}^{d}}: & 627\% & 791\% & 791\% & 1390\% & 1390\% & 19855\%. \end{array} \tag{5.11}$$

These errors show the Gaussian approximation fails to provide reliable results for the noise kernel near the horizon of the optical-Schwarzschild metric. We expect this be the case also for the Schwarzschild metric near the horizon, as we will show next. In the above we have identified that the occurrence of significant error begins at the fourth covariant derivative order.

VI. NOISE KERNEL IN CONFORMALLY OPTICAL SPACETIMES

For a static physical metric g_{ab}, we consider the conformally related optical metric

$$\bar{g}_{ab}=e^{-2\omega}g_ab, \tag{6.1}$$

where the conformal factor $e^{-2\omega}$ is the space-dependent function such that $\bar{g}_{\tau\tau}=1$, i.e., the metric $\bar{g}_{ab}$ is for an ultrastatic spacetime. In general, we use the overbar to denote objects defined in terms of the optical metric $\bar{g}_{ab}$ and we omit it for those in terms of the physical metric g_{ab}. For a conformally invariant field, the Green functions on the two spacetimes are related via

$$G(x,y)=e^{-\omega(x)}\bar{G}(x,y)e^{-\omega(y)}. \tag{6.2}$$

This is our starting point. In terms of the Gaussian approximation

$$\bar{G}_{\rm Gauss}(x,y)=\frac{\kappa\bar{\Delta}^{1/2}}{8\pi^2\bar{r}}\frac{\sinh\kappa\bar{r}}{(\cosh\kappa\bar{r}-\cos\kappa\tau)}, \tag{6.3}$$

where $\bar{r}=[2^{(3)}\bar{\sigma}]^{1/2}$ and κ is the period of the imaginary time dimension, we have the expansion of the physical Green function as

$$G_{\rm Gauss}=\frac{\bar{\Delta}^{1/2}e^{-\omega-\omega'}}{8\bar{\sigma}\pi^2}+\frac{\bar{\Delta}^{1/2}e^{-\omega-\omega'}}{8\pi^2}\Big\{\frac{\kappa^2}{6}+\frac{\kappa^4}{180}(-\bar{\sigma}+2\Delta\tau^2)+\frac{\kappa^6}{3780}(\bar{\sigma}^2-6\bar{\sigma}\Delta\tau^2+4\Delta\tau^4)\Big\}+O(\sigma^{5/2},\Delta\tau^5), \tag{6.4}$$

where $\omega=\omega(x)$ and $\omega'=\omega(y)$.

We proceed as before and subtract from $G_{\rm Gauss}$ the Hadamard ansatz

$$S(x,y)=\frac{1}{16\pi^2}\left(\frac{2\Delta^{1/2}}{\sigma}+\sigma w_1+\sigma^2 w_2\right)+O(\sigma^3) \quad (6.5)$$

to get the renormalized Green function

$$G_{\rm ren}=G_{\rm Gauss}-S. \quad (6.6)$$

In the Hadamard ansatz, the $V(x,x')$ term is not included since there is no $\ln\sigma$ divergence present in the expansion of the Gaussian approximation to the Green function. This can also be viewed as an extension of the Gaussian approximation.

Now the situation here is different from the optical case since the divergent terms present (6.4) and (6.6) do not directly cancel. Much of the work for regularizing the Green function will entail showing that indeed the difference between these two divergent terms is finite and developing this finite difference to sufficient order to compute the noise kernel. To this end, we write the renormalized Green function as

$$G_{\rm ren}=G_{\rm div,ren}+G_{\rm fin}-\frac{1}{(4\pi)^2}W, \quad (6.7)$$

with

$$G_{\rm div,ren}=\frac{1}{8\pi^2}\left(\frac{\bar{\Delta}^{1/2}e^{-\omega-\omega'}}{\bar{\sigma}}-\frac{\Delta^{1/2}}{\sigma}\right), \quad (6.8a)$$

$$G_{\rm fin}=\frac{\bar{\Delta}^{1/2}e^{-\omega-\omega'}}{8\pi^2}\left\{\frac{\kappa^2}{6}+\frac{\kappa^4}{180}(-\bar{\sigma}+2\Delta\tau^2)+\frac{\kappa^6}{3780}(\bar{\sigma}^2-6\bar{\sigma}\Delta\tau^2+4\Delta\tau^4)\right\}, \quad (6.8b)$$

$$W=\sigma w_1+\sigma^2 w_2. \quad (6.8c)$$

Both $G_{\rm fin}$ and W have well behaved coincident limits. As these functions stand, it is only the last one, W, that we can readily handle. Appendix E gives the series expansion in terms of the world function σ defined with respect to the physical metric g_{ab}. Since we want the noise kernel in the physical metric, it is the covariant derivative commensurate with this metric that we must use when computing the noise kernel. On the other hand, the function $G_{\rm fin}$ is defined in terms of the world function and the Van Vleck–Morette determinant of the optical metric $\bar{g}_{ab}$. Thus the coincident limit expressions for σ and the series expansion for $\Delta^{1/2}$ derived in Appendices A and C cannot be used to determine the contribution to a series expansion of the renormalized Green function $G_{\rm ren}$.

We outline ways to get around this problem in Appendix F. Both $\bar{\sigma}$ and $\bar{\Delta}^{1/2}$ are defined in terms of covariant differential equations with respect to the optical metric. The conformal transformation properties of the covariant derivative are used to reexpress these equations in terms of the covariant derivative commensurate with the physical metric. Then end point series solutions, built from the physical metric world function σ are found. The details, along with the found solution, are collected in Appendix F. Using these results, we can determine the contribution from $G_{\rm fin}$ to the noise kernel.

This leaves the first function defined above to address. The key to unlocking this term is to introduce the symmetric function

$$\Sigma(x,y)=e^{\omega(x)}\bar{\sigma}(x,y)e^{\omega(y)}-\sigma(x,y). \quad (6.9)$$

The important property of this function as shown in Appendix F is $\Sigma\sim\sigma^2$ as $\sigma\to 0$. With this function, we have

$$G_{\rm div,ren}=\frac{1}{8\pi^2}\,\frac{\sigma(\bar{\Delta}^{1/2}-\Delta^{1/2})-\Sigma\Delta^{1/2}}{\sigma(\sigma+\Sigma)}. \quad (6.10)$$

To see that this is indeed finite, we use the expansions of

$$\Sigma\sim\sigma^2 S^{(4)}, \quad (6.11a)$$

$$\Delta^{1/2}\sim 1+\sigma\Delta^{(2)}, \quad (6.11b)$$

$$\bar{\Delta}^{1/2}\sim 1+\sigma\bar{\Delta}^{(2)} \quad (6.11c)$$

to obtain the leading order behavior

$$G_{\rm div,ren}=-\frac{1}{8\pi^2}\,\frac{-\bar{\Delta}^{(2)}+\Delta^{(2)}+S^{(4)}+\sigma\Delta^{(2)}S^{(4)}}{1+\sigma S^{(4)}}, \quad (6.12)$$

which is finite as $\sigma\to 0$ and has the value

$$[G_{\rm div,ren}]=\frac{1}{8\pi^2}[\bar{\Delta}^{(2)}-\Delta^{(2)}-S^{(4)}]. \quad (6.13)$$

Using Eqs. (F11a) and (F16a), along with Eq. (C8), we get the explicit form of the leading order expansion tensors used above as

$$\Sigma^{(4)}_{abcd}=\frac{\omega_{;c}\omega_{;d}g_{ab}}{12}+\frac{\omega_{;cd}g_{ab}}{12}-\frac{\omega_{;p}\omega^{;p}g_{ab}g_{cd}}{24}, \quad (6.14a)$$

$$\Delta^{(2)}_{ab}=\frac{R_{ab}}{12}, \quad (6.14b)$$

$$\bar{\Delta}^{(2)}_{ab}=\frac{\omega_{;a}\omega_{;b}}{6}+\frac{\omega_{;ab}}{6}-\frac{\omega_{;p}\omega^{;p}g_{ab}}{6}+\frac{\omega_{;p}{}^{p}g_{ab}}{12}+\frac{R_{ab}}{12}. \quad (6.14c)$$

From these we can get the expansion scalars we need, for Eq. (6.13),

$$S^{(4)}=4p^p p^q p^r p^s\Sigma^{(4)}_{pqrs}=-\frac{(\omega_{;p}\omega^{;p})}{6}+\frac{\omega_{;p}\omega_{;q}p^p p^q}{3}+\frac{\omega_{;pq}p^p p^q}{3}, \quad (6.15a)$$

$$\Delta^{(2)}=2p^pp^q\Delta^{(2)}_{pq}=\frac{p^pp^qR_{pq}}{6}, \tag{6.15b}$$

$$\bar{\Delta}^{(2)}=2p^pp^q\bar{\Delta}^{(2)}_{pq}=-\frac{(\omega_{;p}\omega^{;p})}{3}+\frac{\omega_{;p}{}^{p}}{6}+\frac{\omega_{;p}\omega_{;q}p^pp^q}{3}+\frac{\omega_{;pq}p^pp^q}{3}+\frac{p^pp^qR_{pq}}{6}, \tag{6.15c}$$

where p^a is a unit vector. From these expressions, we might expect there to be residual direction dependence for Eq. (6.13). But when we substitute the expansion scalars into the $\sigma\to 0$ value of $G_{\rm div,ren}$, the direction dependencies of the three expansion scalars cancel and we are left with

$$[G_{\rm div,ren}]=\frac{1}{48\pi^2}(\omega_{;p}{}^{p}-\omega_{;p}\omega^{;p}). \tag{6.16}$$

Using $[\bar{\Delta}^{1/2}]=1$ and $[\bar{\sigma}]=0$, we can immediately get the coincident limit of Eq. (6.8b)

$$[G_{\rm fin}]=\frac{\kappa^2}{48e^{2\omega}\pi^2}. \tag{6.17}$$

Since $[W]=0$, we readily obtain the coincident limit of the renormalized Green function

$$[G_{\rm ren}]=\frac{1}{48\pi^2}\left(\frac{\kappa^2}{e^{2\omega}}-\omega_{;p}\omega^{;p}+\omega_{;p}{}^{p}\right), \tag{6.18}$$

which is the result derived by Page [Eq. (29) of Ref. [47]].

Now that we know we can regularize $G_{\rm div,ren}$, we turn to developing the series expansion of $G_{\rm div,ren}$ to a sufficient order to compute the coincident limit of the noise kernel. We need the expansion

$$G_{\rm div,ren}=\frac{1}{8\pi^2}(G^{(0)}_{\rm div,ren}+\sqrt{\sigma}G^{(1)}_{\rm div,ren}+\sigma G^{(2)}_{\rm div,ren}+\sigma^{3/2}G^{(3)}_{\rm div,ren}+\sigma^2G^{(4)}_{\rm div,ren}). \tag{6.19}$$

The computation of $[G_{\rm div,ren}]$, i.e., $G^{(0)}_{\rm div,ren}$, involves Σ to order σ^2 and both $\Delta^{1/2}$ and $\bar{\Delta}^{1/2}$ to order σ. Thus to carry out the expansion (6.19) we will need these functions expanded to order σ^4 for Σ and σ^3 for $\Delta^{1/2}$ and $\bar{\Delta}^{1/2}$. These are given in Appendix F. For this section we will use expansions in terms of the scalar coefficients:

$$\Sigma\sim\sigma^2S^{(4)}+\sigma^{5/2}S^{(5)}+\sigma^3S^{(6)}+\sigma^{7/2}S^{(7)}+\sigma^4S^{(8)}, \tag{6.20a}$$

$$\Delta^{1/2}\sim 1+\sigma\Delta^{(2)}+\sigma^{3/2}\Delta^{(3)}+\sigma^2\Delta^{(4)}+\sigma^{5/2}\Delta^{(5)}+\sigma^3\Delta^{(6)}, \tag{6.20b}$$

$$\bar{\Delta}^{1/2}\sim 1+\sigma\bar{\Delta}^{(2)}+\sigma^{3/2}\bar{\Delta}^{(3)}+\sigma^2\bar{\Delta}^{(4)}+\sigma^{5/2}\bar{\Delta}^{(5)}+\sigma^3\bar{\Delta}^{(6)}. \tag{6.20c}$$

(Recall that the scalar expansion coefficients are related to the tensor expansion cofficients via $A^{(n)}=2^{n/2}p^{p_1}\cdots p^{p_n}A^{(n)}_{p_1\cdots p_n}$, where in general we take $\sqrt{\sigma}p^a=\sigma^{;a}$.)

Putting these in Eq. (6.10), the expansion coefficients $G^{(n)}_{\rm div,ren}$ are

$$G^{(0)}_{\rm div,ren}=\bar{\Delta}^{(2)}-\Delta^{(2)}-S^{(4)}, \tag{6.21a}$$

$$G^{(1)}_{\rm div,ren}=\bar{\Delta}^{(3)}-\Delta^{(3)}-S^{(5)}, \tag{6.21b}$$

$$G^{(2)}_{\rm div,ren}=\bar{\Delta}^{(4)}-\Delta^{(4)}-\Delta^{(2)}S^{(4)}+S^{(4)}[-\bar{\Delta}^{(2)}+\Delta^{(2)}+S^{(4)}]-S^{(6)}, \tag{6.21c}$$

$$G^{(3)}_{\rm div,ren}=\bar{\Delta}^{(5)}-\Delta^{(5)}-\Delta^{(3)}S^{(4)}-\Delta^{(2)}S^{(5)}+[-\bar{\Delta}^{(2)}+\Delta^{(2)}+S^{(4)}]S^{(5)}+S^{(4)}[-\bar{\Delta}^{(3)}+\Delta^{(3)}+S^{(5)}]-S^{(7)}, \tag{6.21d}$$

$$G^{(4)}_{\rm div,ren}=\bar{\Delta}^{(6)}-\Delta^{(6)}-\Delta^{(4)}S^{(4)}-\Delta^{(3)}S^{(5)}+S^{(5)}[-\bar{\Delta}^{(3)}+\Delta^{(3)}+S^{(5)}]-\{[-\bar{\Delta}^{(2)}+\Delta^{(2)}+S^{(4)}][S^{(4)2}-S^{(6)}]\}-\Delta^{(2)}S^{(6)}+S^{(4)}[-\bar{\Delta}^{(4)}+\Delta^{(4)}+\Delta^{(2)}S^{(4)}+S^{(6)}]-S^{(8)}. \tag{6.21e}$$

We have shown that for $G^{(0)}_{\rm div,ren}$ a direction dependence could arise, but when we use the values of the expansion tensors for Σ, $\Delta^{1/2}$, and $\bar{\Delta}^{1/2}$, this direction dependence cancels. We expect this to happen because we see the coefficients that contribute to any given coefficient of $G_{\rm div,ren}$ are of a higher power than the $G^{(n)}_{\rm div,ren}$ coefficient in question. When going from the scalar expansion coefficient form to the tensor expansion form, the rank of the tensor is the same as the order of the coefficient. And each of the $G^{(n)}_{\rm div,ren}$ is made up of higher order coefficients. But as we find by direct substitution from the expansion tensors listed in Appendix F, for each case the direction dependence always cancels. There we also give the complete form of the coefficients $G^{(n)}_{\rm div,ren}$, in their corresponding tensorial form. This is one of our main results: The regularization of the leading order divergence of the Green function, when the Green function has been computed in an optical metric conformal to the physical metric in which the problem is given. What is new here is that this regularization has been carried out to the order needed for the computation of the noise kernel. This analysis has been developed to take advantage of the symbolic computing potential of current workstations.

The last remaining obstacle in computing the noise kernel comes from $G_{\rm fin}$. As it stands, it is defined in terms of the optical metric, while it is the physical metric with which we need to take the covariant derivatives that determine the noise kernel. By using the function Σ and the series expansions (6.11), along with the series expansions

$$(\delta\tau)^2=\sigma\,\delta\tau^{(2)}+\sigma^{3/2}\delta\tau^{(3)}+\sigma^2\,\delta\tau^{(4)}, \tag{6.22}$$

$$e^{-a(\omega+\omega')}=\omega^{(0,a)}+\sqrt{\sigma}\omega^{(1,a)}+\sigma\omega^{(2,a)}+\sigma^{3/2}\omega^{(3,a)}+\sigma^2\omega^{(4,a)}, \tag{6.23}$$

$G_{\rm fin}$ can be expanded in terms of the physical σ as

$$G_{\rm fin}=\frac{1}{8\pi^2}(G_{\rm fin}^{(0)}+\sqrt{\sigma}G_{\rm fin}^{(1)}+\sigma G_{\rm fin}^{(2)}+\sigma^{3/2}G_{\rm fin}^{(3)}+\sigma^2 G_{\rm fin}^{(4)}), \tag{6.24}$$

where these expansion coefficients are

$$G_{\rm fin}^{(0)}=\frac{\kappa^2}{6}\omega^{(0,1)}, \tag{6.25a}$$

$$G_{\rm fin}^{(1)}=\frac{\kappa^2}{6}\omega^{(1,1)}, \tag{6.25b}$$

$$G_{\rm fin}^{(2)}=\frac{\kappa^2}{6}[\overline{\Delta}^{(2)}\omega^{(0,1)}+\omega^{(2,1)}]+\frac{\kappa^4}{180}[2\,\delta\tau^{(2)}\omega^{(0,1)}-\omega^{(0,2)}], \tag{6.25c}$$

$$G_{\rm fin}^{(3)}=\frac{\kappa^2}{6}[\overline{\Delta}^{(3)}\omega^{(0,1)}+\overline{\Delta}^{(2)}\omega^{(1,1)}+\omega^{(3,1)}]+\frac{\kappa^4}{180}[2\,\delta\tau^{(3)}\omega^{(0,1)}+2\,\delta\tau^{(2)}\omega^{(1,1)}-\omega^{(1,2)}], \tag{6.25d}$$

$$\begin{aligned}G_{\rm fin}^{(4)}=&\frac{\kappa^2}{6}[\overline{\Delta}^{(4)}\omega^{(0,1)}+\overline{\Delta}^{(3)}\omega^{(1,1)}+\overline{\Delta}^{(2)}\omega^{(2,1)}+\omega^{(4,1)}]\\&-\frac{\kappa^4}{180}\{-2\,\delta\tau^{(4)}\omega^{(0,1)}+S^{(4)}\omega^{(0,2)}\\&+\overline{\Delta}^{(2)}[-2\,\delta\tau^{(2)}\omega^{(0,1)}+\omega^{(0,2)}]-2\,\delta\tau^{(3)}\omega^{(1,1)}\\&-2\,\delta\tau^{(2)}\omega^{(2,1)}+\omega^{(2,2)}\}+\frac{\kappa^6}{3780}[4\,\delta\tau^{(2)2}\omega^{(0,1)}\\&-6\,\delta\tau^{(2)}\omega^{(0,2)}+\omega^{(0,3)}].\end{aligned} \tag{6.25e}$$

The explicit expressions for these expansion tensors are given at the end of Appendix F.

With this we have regularized and expanded $G_{\rm div,ren}$, and expanded in the physical metric $G_{\rm fin}$, both to fourth order. The Hadamard ansatz function W's series expansion is derived in Appendix E also to fourth order. From here we proceed as we did for the optical metrics. To review, once a metric is selected, the component values of the expansions of $G_{\rm div,ren}$, $G_{\rm fin}$, and W are computed symbolically on the computer. Then Eq. (3.10) is used to determine the component values of the coincident limit of up to four covariant derivatives of $G_{\rm ren}$, along with the needed covariant derivatives of the coincident limits. From here the component values of the coincident limit of the noise kernel we reached are computed. This procedure is adopted owing to the large size of the expansions. In fact, initial attempts to delay the specification of the metric until a full determination of the noise kernel ended up yielding a general tensorial expression of nearly 60000 terms. By working out from the expansion tensors in terms of their component values, we found this had the added advantage of enabling one to study each of the separate terms that go into computing the noise kernel.

VII. SCHWARZSCHILD BLACK HOLE

We can now turn our attention to a specific example. Consider a massless, conformally coupled scalar field on a Schwarzschild black hole with mass M. The line element is given in the usual coordinates $x^a=(r,\theta,\phi,\tau)$,

$$ds^2=\left(1-\frac{2M}{r}\right)d\tau^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{7.1}$$

where τ is the imaginary time in the Euclidean sector of the spacetime. With the conformal factor

$$e^{2w}=1-\frac{2M}{r}, \tag{7.2}$$

this metric is the physical metric corresponding to the optical metric considered in the above. As in that case, the imaginary time dimension has periodicity $\kappa=1/4M$, corresponding to a temperature associated with the Hartle-Hawking state. We use the scaled spatial coordinate $x\equiv 2M/r\equiv 1/2\kappa r$. With this choice, the black hole horizon $r=2M$ is at $x=1$ while spatial infinity is at $x=0$.

We now use the results for the expansion tensors above to compute the noise kernel coincident limit, along with the stress tensor itself. For the stress tensor,

$$\begin{aligned}\langle T_a{}^b\rangle=(\rho_\infty){\rm diag}\{&(1+2x+3x^2+4x^3+5x^4+6x^5+15x^6)/3,(1+2x+3x^2)(1+4x^3-3x^4)/3,(1+2x\\&+3x^2)(1+4x^3-3x^4)/3,-1-2x-3x^2-4x^3-5x^4-6x^5+33x^6\},\end{aligned} \tag{7.3}$$

where

$$\rho_\infty=\frac{\kappa^4}{480\pi^2}=-\langle T_\tau{}^\tau\rangle|_{r\to\infty}. \tag{7.4}$$

This result agrees with Page's Eq. (83) [47]. In his work, Page showed the stress tensor must satisfy a functional-differential scale equation under conformal transformations. He then found, via trial and error, a general solution to this equation. This became the basis for his computation of the stress tensor in the black hole metric.

In contrast, we have worked directly with the Green function and the point-separation definition of the stress tensor. The conformal transformation properties of the geometric objects that go into the Green function are studied and the corresponding series expansions are developed. As can be seen from Appendix F these expansions can get quite involved even for just up to the second order terms needed for the stress tensor. Thus our agreement with Page on the stress tensor serves as an anchor for this work. Moreover, the methods used for determining the series expansions are fundamentally recursive. The zero order result (6.18) and the second order result (7.3) are in agreement with known results. Since the terms needed for the noise kernel are generated recursively from these the symbolically computed results for the Schwarzschild noise kernel coincident limit should be accurate, up to the validity of the Gaussian approximation to the Green function. These results are

$$N_{\tau}{}^{\tau}{}_{\tau}{}^{\tau}=\frac{{\rho_\infty}^2}{756}(219+876x+2190x^2+4380x^3+7215x^4 +10464x^5++16920x^6+21424x^7-2984943x^8 +219140x^9+197314x^{10}+180260x^{11} +3292493x^{12}), \tag{7.5a}$$

$$N_{r}{}^{r}{}_{r}{}^{r}=\frac{{\rho_\infty}^2}{2268}(137+548x+1370x^2+3860x^3+9005x^4 +98432x^5+225080x^6+408752x^7+2553371x^8 +1725900x^9+1206822x^{10}+761100x^{11} -4090521x^{12}), \tag{7.5b}$$

$$N_{\theta}{}^{\theta}{}_{\theta}{}^{\theta}=\frac{{\rho_\infty}^2}{2268}(137+548x+1370x^2++2180x^3+3965x^4 +37952x^5+77240x^6+102992x^7-11973349x^8 +1817100x^9+1721382x^{10}+1630140x^{11} +12756759x^{12}), \tag{7.5c}$$

$$N_{\tau}{}^{\tau}{}_{r}{}^{r}=\frac{{\rho_\infty}^2}{2268}(-219-876x-2190x^2-2140x^3-495x^4 +2976x^5-10984x^6-49872x^7+1327551x^8 -230916x^9-50834x^{10}+95356x^{11} -774845x^{12}), \tag{7.5d}$$

$$N_{\tau}{}^{\tau}{}_{\theta}{}^{\theta}=\frac{{\rho_\infty}^2}{2268}(-219-876x-2190x^2-5500x^3 -10575x^4-17184x^5-19888x^6-7200x^7 -10917561x^8+1056828x^9+999526x^{10} +952012x^{11}+11087083x^{12}), \tag{7.5e}$$

$$N_{r}{}^{r}{}_{\theta}{}^{\theta}=\frac{{\rho_\infty}^2}{2268}(41+164x+410x^2-860x^3-4255x^4 -50704x^5-107048x^6-179440x^7+1023059x^8 +159708x^9+329206x^{10}+478972x^{11} +1465003x^{12}). \tag{7.5f}$$

Knowing the component values, we determine the trace to be

$$N_{p}{}^{p}{}_{q}{}^{q}=\frac{32000{\rho_\infty}^2x^8}{21}(1+x)(-27+31x)(1+x^2). \tag{7.6}$$

As with the optical-Schwarzschild case, the trace under the Gaussian approximation fails to vanish, which it should, for the massless conformal coupling case we are considering. By taking the trace of the coincident limit expressions for the noise kernel above, term by term, we find that this arises from the nonvanishing of the fourth order derivative terms such as $[G_{\mathrm{ren};p}{}^{p}{}_{q}{}^{q}]$. This is what we had expected, based on our analysis of the optical-Schwarzschild metric. It is not from our implementation of the conformal transformation.

Our result yields a zero trace at spatial infinity $x\to 0$ or $r\to\infty$. The fluctuation measure Δ_{abcd} at $r\to\infty$ gives the hot flat space result obtained before in Eqs. (4.7) and (5.10). At the horizon the magnitude of the error is

$$\begin{array}{lcccccc} abcd: & \tau\tau\tau\tau & rrrr & \theta\theta\theta\theta & \tau\tau rr & \tau\tau\theta\theta & rr\theta\theta, \\ \dfrac{N_{p}{}^{p}{}_{q}{}^{q}}{N_{a}{}^{b}{}_{c}{}^{d}}: & 1904\% & 1904\% & 894\% & 18278\% & 1775\% & 1775\%. \end{array} \tag{7.7}$$

These results show that the Gaussian approximation has completely broken down at the horizon.

This is not to say that we can draw no conclusions other than to discover the inadequacies of the Gaussian approximation. What is important is the finiteness at the horizon of both the noise kernel components and the error as expressed by the failure of the trace of the noise kernel to vanish. In our computation of both the stress tensor and the noise kernel, we have discovered that finiteness at the horizon is a ‘‘fragile’’ property. By this we mean that any small error in the symbolic code would result in a noise kernel that diverges as $x\to 1$. This has lead us to develop more than one way to determine the series for optical metric geometric objects, just to test the symbolic code. We arrive at the results (7.6) using more than one computational path. In contrast if there were one single error in the code, the resulting noise kernel could it be finite on the horizon.

It is the noise kernel itself, via its trace, that provides a measure of the error of the Gaussian approximation, i.e., we self-consistently compute both the noise kernel and its error. Any scheme to correct or improve the error of the Gaussian approximation should also apply to the noise kernel. Correcting the Gaussian approximation will amount to finding the terms that need to be included such that they satisfy the field equation to fourth order in σ^a. Any correction one uses to compute the noise kernel (7.5f) will have to exactly cancel the current trace (7.6). Hence the correction to the noise kernel will itself be finite at the horizon. With this in mind, we can conclude that the fluctuations of the stress tensor, as measured by the coincident limit of the noise kernel, are finite at the black hole's horizon. This is one of the main lessons learned from the analysis of the noise kernel via point separation.

VIII. DISCUSSIONS

Let us summarize our findings pertaining to two sets of issues: the range of validity of the Gaussian approximation and the results and usefulness of our program in spite of this approximation.

Despite its success for the stress tensor calculation there is no compelling reason to expect the Gaussian approximated Green function to produce reasonable results for expressions involving higher order covariant derivatives, such as these in the noise kernel. Nonetheless the Gaussian approximation is very relevant because it contains the leading order divergence. This structure will remain even with a better approximation, while this leading order divergence must be regularized. This step is needed regardless of what form of the Green function one adopts—whether future improvements to the Gaussian approximation remain desirable or the exact Green function is derived in the optical metric only. Our work lays down the structure and provides the details for its implementation.

Now we present the successes and failures of the Gaussian approximation as applied to the computation of the noise kernel. On the positive side our results for the fluctuations of the stress tensor of the Hawking flux in the far field region agrees with the analytic results of Campos and Hu [8]. A fringe benefit is that we can verify our procedure by explicitly rederiving the Page result [47] for the stress tensor. We note that in Page's original work, the direct use of the conformal transformation was circumvented by ''guessing'' the solution to a functional-differential equation. Our result is the first we know where the methodology of point separation was carried all the way through to the final result. That we get the known results is a confirmation of our method and its correct implementation. On the negative side, our calculation shows that the fluctuations of the stress tensor based on the Gaussian approximation are unreliable in regions close to the event horizon. We show this by checking that the trace anomaly fails to vanish there. Corrections to the Gaussian approximation need be introduced to improve the accuracy. One important result we found (which may be overshadowed by the ''glaring'' error of the Gaussian approximation) is a *finite* expression for the noise kernel. Even though the error is large, as long as it is finite, we know corrections to the approximations will themselves be finite. Hence, where the approximate noise kernel is finite, we can expect the full noise kernel to be finite. That the noise kernel on the horizon of a Schwarzschild black hole is finite is in itself a qualitatively significant result. This dispels claims to the contrary based on speculative arguments or less than rigorous calculations [18–21].

Along the way to regularizing the Green function to fourth order, necessary for the coincident limit of the noise kernel, we have developed the series expansions of the various geometric objects that make up the Green function to a high order. Once the additional terms necessary to correct the Gaussian approximation are determined, it is a simple matter to use the work contained herein to compute these correction terms to a high enough order. In this sense, this work not only lays down the tracks and defines the steps, but also provides all the details necessary for implementing the point-separation program to calculate the regularized noise kernel for quantum fields in curved spacetime. This is useful for, e.g., tackling the black hole fluctuations and back reaction problem [25,26].

In conclusion, towards the three goals set for this work, we have detailed the steps in the calculation of the regularized noise kernel using the modified point-separation scheme under the Gaussian approximation for the Green function of a quantum scalar field in optical and conformally optical spacetimes. We have derived the regularized noise kernel for a thermal field in flat space and for a quantum field in optical-Schwarzschild spacetime. We have obtained a finite expression for the noise kernel at the horizon of a Schwarzschild spacetime and recovered the hot flat space result at infinity. From the error in the noise kernel at the horizon we showed that the Gaussian approximation scheme of Bekenstein-Parker-Page applied to the Green function, which provides surprisingly good results for the stress tensor (involving the second covariant derivative order of the Green function). fails at the fourth covariant derivative order.

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APPENDIX A: WORLD FUNCTION σ

In this appendix, we review the properties of the world function σ. We also demonstrate how symbolic computations are implemented and used in this work. Christensen's [43] method for determining the coincident limit of covariant derivatives of functions defined via a covariant differential equation is reviewed.

The world function is defined to be one-half of the square of the geodesic distance between two points on a differential manifold. As such, it satisfies the equation

$$\sigma - \frac{\sigma_{;p}\sigma^{;p}}{2} = 0 \tag{A1}$$

along with the initial value

$$[\sigma_{;a}] = 0. \tag{A2}$$

To determine $[\sigma_{;ab}]$, we take two derivatives of Eq. (A1). To help illustrate how these calculations are done on the computer, the output presented here is direct output from MATHTENSOR [50], with MATHEMATICA [51] carrying out the formatting. We use the MATHTENSOR function compact disc (CD) and get

$$\sigma_{;ab}-\frac{\sigma_{;pb}\sigma_{;a}{}^{p}}{2}-\frac{\sigma_{;pa}\sigma_{;b}{}^{p}}{2}-\frac{\sigma^{;p}\sigma_{;pab}}{2}-\frac{\sigma_{;p}\sigma_{;a}{}^{p}{}_{b}}{2}=0. \tag{A3}$$

This is put into a canonical form via Canonicalize,

$$\sigma_{;ab}-\sigma_{;pa}\sigma_{;b}{}^{p}-\sigma_{;p}\sigma_{;a}{}^{p}{}_{b}=0. \tag{A4}$$

The condition (A2) is encoded into MATHEMATICA by defining a rule Ci[CD[sigma,la_]]->0, where Ci is a function defined to represent the coincident limit and is formatted to be displayed using the standard $[\cdots]$ notation. Using this rule, the coincident limit is

$$[\sigma_{;ab}]-[\sigma_{;pa}][\sigma_{;b}{}^{p}]=0. \tag{A5}$$

This immediately shows $[\sigma_{;ab}]=g_{ab}$ and such a rule is defined.

Proceeding with one more covariant derivative,

$$0=\sigma_{;abc}-\sigma_{;pc}\sigma_{;a}{}^{p}{}_{b}-\sigma_{;pb}\sigma_{;a}{}^{p}{}_{c}-\sigma_{;pa}\sigma_{;b}{}^{p}{}_{c}-\sigma_{;p}\sigma_{;a}{}^{p}{}_{bc}, \tag{A6}$$

and using the two rules we already have, we recursively get

$$0=[\sigma_{;abc}]+[\sigma_{;acb}]. \tag{A7}$$

Using MATHTENSORS' OrderCD, which commutes the covariant derivatives on each term until they are in alphabetical order, we get the result

$$0=2[\sigma_{;abc}]+[\sigma_{;p}]R_{a}{}^{p}{}_{bc}=2[\sigma_{;abc}] \tag{A8}$$

where Eq. (A2) is used to go from the first to the second line. We have that the coincident limit of three covariant derivatives acting on σ vanishes.

Proceeding with one more covariant derivative we get

$$\begin{aligned}0=&\sigma_{;pad}\sigma_{;b}{}^{p}{}_{c}+\sigma_{;pac}\sigma_{;b}{}^{p}{}_{d}+\sigma_{;pab}\sigma_{;c}{}^{p}{}_{d}-\sigma_{;abcd}\\&+\sigma_{;pd}\sigma_{;a}{}^{p}{}_{bc}+\sigma_{;pc}\sigma_{;a}{}^{p}{}_{bd}+\sigma_{;pb}\sigma_{;a}{}^{p}{}_{cd}+\sigma_{;pa}\sigma_{;b}{}^{p}{}_{cd}\\&+\sigma_{;p}\sigma_{;a}{}^{p}{}_{bcd}\Rightarrow 0=[\sigma_{;abcd}]+[\sigma_{;acbd}]+[\sigma_{;adbc}],\end{aligned} \tag{A9}$$

where once again we use the rules we already know. Now, commuting the covariant derivatives

$$\begin{aligned}0=&3[\sigma_{;abcd}]+(R_{a}{}^{p}{}_{bc;d}+R_{a}{}^{p}{}_{bd;c})[\sigma_{;p}]+[\sigma_{;pd}]R_{a}{}^{p}{}_{bc}\\&+[\sigma_{;pc}]R_{a}{}^{p}{}_{bd}+[\sigma_{;pb}]R_{a}{}^{p}{}_{cd}+[\sigma_{;pa}]R_{b}{}^{p}{}_{cd}\end{aligned} \tag{A10}$$

and using the known rules, we get the equation

$$0=3[\sigma_{;abcd}]+R_{acbd}+R_{adbc} \tag{A11}$$

which we can solve for the term we need. In practice, MATHEMATICA's function Solve[ ] is used, giving us the rule

$$[\sigma_{;abcd}]\rightarrow-\frac{1}{3}(R_{acbd}+R_{adbc}). \tag{A12}$$

By knowing the coincident limit of $n-1$ covariant derivatives of σ, we determine the coincident limit of n covariant derivatives. This is the recursive algorithm developed by Christensen. It is the main idea we use for computing the expansions needed in this work. A general outline starts by assuming we have the rules for $n-1$ covariant derivatives, then, (1) we take n covariant derivatives of the defining equation [in this case, Eq. (A1)]; (2) use the rules for $n-1$ covariant derivatives to get the coincident limit; (3) commute the covariant derivatives; (4) use again the rules for $n-1$ covariant derivatives on the terms generated; (5) solve for the coincident limit of the n derivative term; (6) define a new rule for this term. These steps are iterated until all terms needed are generated.

For the world function σ, we need to carry this out to eight covariant derivatives. The seventh and eighth order (derivative) terms become quite large. In fact, when computing these expressions, we only substitute [steps (2) and (4) above] up to four covariant derivatives and still get results with 240 terms for the seven derivative result and 1101 for the eight. We only finish carrying out the recursion when we use these highest order values.

The results for five and six covariant derivatives are

$$[\sigma_{;abcde}]=-\frac{1}{4}(R_{acbd;e}+R_{acbe;d}+R_{adbc;e}+R_{adbe;c}+R_{aebc;d}+R_{aebd;c}), \tag{A13}$$

$$\begin{aligned}[\sigma_{;abcdef}]=&-(R_{acbd;ef}+R_{acbe;df}+R_{acbf;de}+R_{adbc;ef}+R_{adbe;cf}+R_{adbf;ce}+R_{aebc;df}+R_{aebd;cf}+R_{aebf;cd}+R_{afbc;de}\\&+R_{afbd;ce}+R_{afbe;cd})/5-(R_{pecd}R_{abf}{}^{p}+R_{pebd}R_{acf}{}^{p}+R_{pebc}R_{adf}{}^{p}+R_{pead}R_{bcf}{}^{p}+R_{peac}R_{bdf}{}^{p}+R_{peab}R_{cdf}{}^{p})/9\\&+(R_{pdcf}R_{abe}{}^{p}+R_{pdce}R_{abf}{}^{p}+R_{pdbf}R_{ace}{}^{p}+R_{pdbe}R_{acf}{}^{p}+R_{pdaf}R_{bce}{}^{p}+R_{pdae}R_{bcf}{}^{p})/45+[R_{pcbf}(R_{ade}{}^{p}+7R_{aed}{}^{p})\\&+R_{pcbe}(R_{adf}{}^{p}+7R_{afd}{}^{p})+R_{pcbd}(R_{aef}{}^{p}+7R_{afe}{}^{p})+R_{pcaf}(R_{bde}{}^{p}+7R_{bed}{}^{p})+R_{pcae}(R_{bdf}{}^{p}+7R_{bfd}{}^{p})\\&+R_{pcad}(R_{bef}{}^{p}+7R_{bfe}{}^{p})]/45+\{-[R_{pdbc}(5R_{aef}{}^{p}-R_{afe}{}^{p})]-R_{pdac}(5R_{bef}{}^{p}-R_{bfe}{}^{p})+2R_{pbaf}(5R_{cde}{}^{p}-R_{ced}{}^{p})\\&+2R_{pbae}(5R_{cdf}{}^{p}-R_{cfd}{}^{p})+2R_{pbad}(5R_{cef}{}^{p}-R_{cfe}{}^{p})-R_{pdab}(5R_{cef}{}^{p}-R_{cfe}{}^{p})+2R_{pbac}(5R_{def}{}^{p}-R_{dfe}{}^{p})\\&-R_{pcab}(5R_{def}{}^{p}-R_{dfe}{}^{p})\}/45.\end{aligned} \tag{A14}$$

APPENDIX B: END POINT SERIES EXPANSION

The basic input into the computation of the stress tensor or noise kernel is the Green function, a perfect example of a biscalar. We want to express it in such a way that we can easily identify how it depends on the distance between its two support points. This leads us to consider *series expansions* of biscalars. The techniques are also useful for the series expansions of bitensors.

The world function σ introduced above provides the ideal geometric object for such a construction. It contains both distance and direction information. For a biscalar $S(x,y)$, the *end point expansion* is

$$S(x,y)=A^{(0)}+\sigma^p A^{(1)}_p+\sigma^p\sigma^q A^{(2)}_{pq}+\cdots+\sigma^{p_1}\cdots\sigma^{p_n}A^{(n)}_{p_1\cdots p_n}+\cdots, \tag{B1}$$

so called because the expansion tensors $A^{(n)}_{a_1\cdots a_n}=A^{(n)}_{a_1\cdots a_n}(x)$ have support at one of the end points for which $S(x,y)$ has support.

It is only the symmetric part of the expansion tensors $A^{(n)}_{a_1\cdots a_n}$ that contributes to the expansion, since they are contracted against symmetric products of σ^{p_i}'s. Moreover, the expansion tensors are order n in distance contribution to the biscalar $S(x,y)$. We also find it convenient to have an expansion where the distance dependence is separated from the direction dependence. To this end, if p^a is the unit vector along the geodesic from x to y, $\sigma^a=(2\sigma)^{1/2}p^a$, the expansion (B1) can be reexpressed as

$$S(x,y)=A^{(0)}+\sigma^{1/2}A^{(1)}+\sigma A^{(2)}+\cdots+\sigma^{n/2}A^{(n)}+\cdots, \tag{B2}$$

where $A^{(n)}=2^{n/2}p^{p_1}\cdots p^{p_n}A^{(n)}_{p_1\cdots p_n}$. Now the expansion scalars $A^{(n)}$ carry the direction information.

When multiplying series, this form readily collects terms by their order in distance. In the context of symbolic manipulation of series on the computer, this alternate form greatly improves processing speed.

The first series expansion to consider is that for the world function, which, by virtue of its defining differential equation, is given by

$$\sigma(x,y)=\frac{g_{pq}}{2}\sigma^p\sigma^q. \tag{B3}$$

This is exact and from it we can see that all expansion tensors of order $n\geqslant 3$ vanish. This in turn tells us that $[\sigma_{;(a_1a_2a_3\cdots a_n)}]=0$, i.e., the coincident limit of three or more symmetrized covariant derivatives of the world function vanish. This can also be seen by direct inspection of the previously given expressions for the coincident limits.

We now turn to relating the expansion tensors to the coincident limits of the derivatives of the scalar S. This is done by taking covariant derivatives and then the coincident limit of Eq. (B1). We immediately get

$$[S]=A^{(0)}. \tag{B4}$$

Taking one derivative and the coincident limit gives

$$[S_{;a}]=A^{(1)}{}_a+A^{(0)}{}_{;a}\Rightarrow A^{(1)}{}_a=-[S]_{;a}+[S_{;a}]. \tag{B5}$$

Two covariant derivatives yield

$$[S_{;ab}]=A^{(2)}{}_{ab}+A^{(2)}{}_{ba}+A^{(1)}{}_{a;b}+A^{(1)}{}_{b;a}+A^{(0)}{}_{;ab}. \tag{B6}$$

We only need the symmetric part of $A^{(2)}{}_{ab}$, or if we assume $A^{(2)}{}_{ab}$ is symmetric, we solve for

$$A^{(2)}{}_{ab}=\frac{1}{2}(-[S_{;a}]_{;b}-[S_{;b}]_{;a}+[S]_{;ab}+[S_{;ab}]). \tag{B7}$$

We write this as

$$A^{(2)}{}_{ab}\doteq-[S_{;a}]_{;b}+\frac{[S]_{;ab}}{2}+\frac{[S_{;ab}]}{2}, \tag{B8}$$

where we use the standard notation $\doteq$ to denote equality upon symmetrization. In terms of symbolic processing, this is implemented by taking each term of a tensorial expression and putting all free indices in lexigraphic order. We also define rules that set to zero any Riemann curvature tensor R_{abcd} when either the first or second pair of indices is free. For example, consider when we take three covariant derivatives:

$$\begin{aligned}[S_{;abc}]=&A^{(3)}{}_{abc}+A^{(3)}{}_{acb}+A^{(3)}{}_{bac}+A^{(3)}{}_{bca}+A^{(3)}{}_{cab}\\&+A^{(3)}{}_{cba}+A^{(2)}{}_{ab;c}+A^{(2)}{}_{ac;b}+A^{(2)}{}_{ba;c}+A^{(2)}{}_{bc;a}\\&+A^{(2)}{}_{ca;b}+A^{(2)}{}_{cb;a}+A^{(1)}{}_{a;bc}+A^{(1)}{}_{b;ac}+A^{(1)}{}_{c;ab}\\&+\frac{A^{(1)}{}_pR_{abc}{}^p}{3}+\frac{A^{(1)}{}_pR_{acb}{}^p}{3}+A^{(0)}{}_{;abc}.\end{aligned} \tag{B9}$$

Now, putting all free indices in lexigraphic order and then using the above rules for the Riemann curvature tensor gives

$$\begin{aligned}[S_{;abc}]\doteq&6A^{(3)}{}_{abc}+6A^{(2)}{}_{ab;c}+3A^{(1)}{}_{a;bc}+A^{(0)}{}_{;abc}\\&+\frac{2A^{(1)}{}_pR_{abc}{}^p}{3}\\\doteq&6A^{(3)}{}_{abc}+6A^{(2)}{}_{ab;c}+3A^{(1)}{}_{a;bc}+A^{(0)}{}_{;abc}.\end{aligned} \tag{B10}$$

This can now be solved for $A^{(3)}{}_{abc}$ and the previously determined results for $A^{(0)}$, $A^{(1)}{}_a$, and $A^{(2)}{}_{ab}$ used to get $A^{(3)}{}_{abc}$ solely in terms of the coincident limit of up to three covariant derivatives acting on S. Nothing new is encountered when determining the rest of the expansion tensors. We now give the results for up through $A^{(8)}{}_{abcdefgh}$:

$$A^{(3)}{}_{abc}\doteq\frac{1}{6}[S_{;abc}]-\frac{1}{2}[S_{;ab}]_{;c}+\frac{1}{2}[S_{;a}]_{;bc}-\frac{1}{6}[S]_{;abc}, \tag{B11a}$$

$$A^{(4)}{}_{abcd} \doteq \frac{1}{24}[S_{;abcd}] - \frac{1}{6}[S_{;abc}]_{;d} + \frac{1}{4}[S_{;ab}]_{;cd} - \frac{1}{6}[S_{;a}]_{;bcd} + \frac{1}{24}[S]_{;abcd}, \tag{B11b}$$

$$A^{(5)}{}_{abcde} \doteq \frac{1}{120}[S_{;abcde}] - \frac{1}{24}[S_{;abcd}]_{;e} + \frac{1}{12}[S_{;abc}]_{;de} - \frac{1}{12}[S_{;ab}]_{;cde} + \frac{1}{24}[S_{;a}]_{;bcde} - \frac{1}{120}[S]_{;abcde}, \tag{B11c}$$

$$A^{(6)}{}_{abcdef} \doteq \frac{1}{720}[S_{;abcdef}] - \frac{1}{120}[S_{;abcde}]_{;f} + \frac{1}{48}[S_{;abcd}]_{;ef} - \frac{1}{36}[S_{;abc}]_{;def} + \frac{1}{48}[S_{;ab}]_{;cdef} - \frac{1}{120}[S_{;a}]_{;bcdef} + \frac{1}{720}[S]_{;abcdef}, \tag{B11d}$$

$$A^{(7)}{}_{abcdefg} \doteq \frac{1}{5040}[S_{;abcdefg}] - \frac{1}{720}[S_{;abcdef}]_{;g} + \frac{1}{240}[S_{;abcde}]_{;fg} - \frac{1}{144}[S_{;abcd}]_{;efg} + \frac{1}{144}[S_{;abc}]_{;defg} - \frac{1}{240}[S_{;ab}]_{;cdefg} + \frac{1}{720}[S_{;a}]_{;bcdefg} - \frac{1}{5040}[S]_{;abcdefg}, \tag{B11e}$$

$$A^{(8)}{}_{abcdefgh} \doteq \frac{1}{40320}[S_{;abcdefgh}] - \frac{1}{5040}[S_{;abcdefg}]_{;h} + \frac{1}{1440}[S_{;abcdef}]_{;gh} - \frac{1}{720}[S_{;abcde}]_{;fgh} + \frac{1}{576}[S_{;abcd}]_{;efgh} - \frac{1}{720}[S_{;abc}]_{;defgh} + \frac{1}{1440}[S_{;ab}]_{;cdefgh} - \frac{1}{5040}[S_{;a}]_{;bcdefgh} + \frac{1}{40320}[S]_{;abcdefgh}. \tag{B11f}$$

These relations simplify considerable if the scalar S is symmetric, $S(x,y)=S(y,x)$, because the symmetrized odd derivatives are determined by the even derivatives. For one derivative,

$$S(x,y)_{;a'} = S(y,x)_{;a'} \Rightarrow [S_{;a'}] = [S_{;a}], \tag{B12}$$

and applying Synge's theorem to the left hand side above yields

$$[S]_{;a} - [S_{;a}] = [S_{;a}] \Rightarrow [S_{;a}] = \frac{[S]_{;a}}{2} \tag{B13}$$

For three derivatives, we have

$$[S_{;a'b'c'}] = [S_{;abc}] \tag{B14}$$

Once again, using Synge's theorem and Eq. (B13),

$$[S_{;ab}]_{;c} + [S_{;ac}]_{;b} + [S_{;bc}]_{;a} - \frac{[S]_{;acb}}{2} - [S_{;bca}] = [S_{;abc}]. \tag{B15}$$

It follows from this that

$$4[S_{;(abc)}] = 6[S_{;(ab}]_{;c)} - [S]_{;(abc)}. \tag{B16a}$$

The results for five and seven derivatives are

$$2[S_{;(abcde)}] = 5[S_{;(abcd}]_{;e)} - 5[S_{;(ab}]_{;cde)} + [S]_{;(abcde)} \tag{B16b}$$

$$8[S_{;(abcdefg)}] = 28[S_{;(abcdef}]_{;g)} - 70[S_{;(abcd}]_{;efg)} + 84[S_{;(ab}]_{;cdefg)} - 17[S]_{;(abcdefg)}. \tag{B16c}$$

With these results, the equations for the even expansion tensors for a symmetric function simplify:

$$A^{(0)} = [S], \tag{B17a}$$

$$2!A^{(2)}{}_{ab} = [S_{;ab}], \tag{B17b}$$

$$4!A^{(4)}{}_{abcd} \doteq [S_{;abcd}], \tag{B17c}$$

$$6!A^{(6)}{}_{abcdef} \doteq [S_{;abcdef}], \tag{B17d}$$

$$8!A^{(8)}{}_{abcdefgh} \doteq [S_{;abcdefgh}], \tag{B17e}$$

while the odd expansion tensors are given in terms of the even tensors:

$$2!A^{(1)}{}_{a} \doteq -A^{(0)}{}_{;a}, \tag{B18a}$$

$$4!A^{(3)}{}_{abc} \doteq -12A^{(2)}{}_{ab;c} + A^{(0)}{}_{;abc}, \tag{B18b}$$

$$6!A^{(5)}{}_{abcde} \doteq -360A^{(4)}{}_{abcd;e} + 30A^{(2)}{}_{ab;cde} - 3A^{(0)}{}_{;abcde}, \tag{B18c}$$

$$8!A^{(7)}{}_{abcdefg} \doteq -20160A^{(6)}{}_{abcdef;g} + 1680A^{(4)}{}_{abcd;efg} - 168A^{(2)}{}_{ab;cdefg} + 17A^{(0)}{}_{;abcdefg}. \tag{B18d}$$

Very often, we need the covariant derivative of a series expansion (B1):

$$S_{;a}=A^{(0)}{}_{;a}+\sigma^{;p}A^{(1)}{}_{p;a}+A^{(1)}{}_{p}\sigma_{;a}{}^{p}+\sigma^{;p}\sigma^{;q}A^{(2)}{}_{pq;a}+A^{(2)}{}_{pq}\sigma^{;q}\sigma_{;a}{}^{p}+A^{(2)}{}_{pq}\sigma^{;p}\sigma_{;a}{}^{q}\cdots. \quad \text{(B19)}$$

If we replace $\sigma_{;ab}$ with its series expansion, then we readily have the series expansion of $S_{;a}$. We can get this via the above relations by merely replacing S with $\sigma_{;ab}$. In particular, we want the expansion

$$\sigma_{;ab}=B^{(0)}{}_{ab}+B^{(1)}{}_{abp}\sigma^{;p}+B^{(2)}{}_{abpq}\sigma^{;p}\sigma^{;q}+B^{(3)}{}_{abpqr}\sigma^{;p}\sigma^{;q}\sigma^{;r}+B^{(4)}{}_{abpqrs}\sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}+B^{(5)}{}_{abpqrst}\sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}\sigma^{;t}+B^{(6)}{}_{abpqrstu}\sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}\sigma^{;t}\sigma^{;u}. \quad \text{(B20)}$$

We immediately have

$$B^{(0)}{}_{ab}=[\sigma_{;ab}]=g_{ab}, \quad \text{(B21)}$$

$$B^{(1)}{}_{abc}=-[\sigma_{;ab}]_{;c}+[\sigma_{;abc}]=0. \quad \text{(B22)}$$

For the second order term,

$$B^{(2)}{}_{abcd}=-[\sigma_{;abc}]_{;d}+\frac{[\sigma_{;ab}]_{;cd}}{2}+\frac{[\sigma_{;abcd}]}{2}=-\frac{1}{6}(R_{acbd}+R_{adbc}). \quad \text{(B23)}$$

Now we have to be careful about how we carry out the symmetrization: it is only the indices c,d that are contracted over in the series (B20). So it is only the free indices *other than a and b* in this and the following that we put in lexographic order (our routine for ordering free indices can be given a list of indices to exclude from ordering):

$$B^{(2)}{}_{abcd}\doteq'-\frac{R_{acbd}}{3}. \quad \text{(B24)}$$

Equality upon symmetrization of all indices but a,b is denoted by $\doteq'$. The rest of the expansion tensors are computed in the same way; the results are

$$B^{(3)}{}_{abcde}\doteq'\frac{R_{acbd;e}}{12}, \quad \text{(B25a)}$$

$$B^{(4)}{}_{abcdef}\doteq'\frac{1}{180}(-3R_{acbd;ef}+4R_{pcad}R_{bef}{}^{p}), \quad \text{(B25b)}$$

$$B^{(5)}{}_{abcdefg}\doteq'\frac{1}{360}(R_{acbd;efg}-3R_{pdbc;e}R_{agf}{}^{p}-3R_{pdac;e}R_{bgf}{}^{p}), \quad \text{(B25c)}$$

$$B^{(6)}{}_{abcdefgh}\doteq'\frac{1}{15120}(-219R_{pdac;e}R_{bfg}{}^{p}{}_{;h}-6R_{acbd;efgh}-114R_{pdbc;ef}R_{agh}{}^{p}-114R_{pdac;ef}R_{bgh}{}^{p}+24R_{peaf}R_{qhbg}R_{c}{}^{p}{}_{d}{}^{q}+8R_{peaf}R_{qhbg}R_{c}{}^{q}{}_{d}{}^{p}). \quad \text{(B25d)}$$

APPENDIX C: VAN VLECK–MORETTE DETERMINANT

Other than the world function, the other main geometric object we need is the Van Vleck–Morette determinant, defined as

$$D(x,y)\equiv-\det(-\sigma_{;ab'}). \quad \text{(C1)}$$

In the context of the Green function, what appears is

$$\Delta^{1/2}(x,y)=\left(\frac{D(x,y)}{\sqrt{g(x)g(y)}}\right)^{1/2}, \quad \text{(C2)}$$

upon which we focus. Using $2\sigma=\sigma^{;p}\sigma_{;p}$ the Van Vleck–Morette determinant is seen to satisfy

$$D^{-1}(D\sigma^{;p})_{;p}=4\Rightarrow\Delta^{1/2}(4-\Box\sigma)-2\Delta^{1/2}_{;p}\sigma^{;p}=0 \quad \text{(C3)}$$

along with

$$[D]=g(x)\Rightarrow[\Delta^{1/2}]=1, \quad \text{(C4)}$$

from which we readily get

$$[\Delta^{1/2}_{;a}]=0. \quad \text{(C5)}$$

We could at this point proceed as we did with σ to determine the coincident limit expression of covariant derivatives of $\Delta^{1/2}$. But what we need is the end point expansion of $\Delta^{1/2}(x,y)$ to sixth order in σ^{a}. With this in mind, we set out to directly determine the series. We start by assuming the expansion,

$$\Delta^{1/2}=1+\Delta^{(2)}_{pq}\sigma^{p}\sigma^{q}+\Delta^{(3)}_{pqr}\sigma^{p}\sigma^{q}\sigma^{r}+\Delta^{(4)}_{pqrs}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}+\Delta^{(5)}_{pqrst}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}\sigma^{t}+\Delta^{(6)}_{pqrstu}\sigma^{p}\sigma^{q}\sigma^{r}\sigma^{s}\sigma^{t}\sigma^{u} \quad \text{(C6)}$$

and substitute back into Eq. (C3). We use Eq. (B20) and $\sigma^{a}=(2\sigma)^{1/2}p^{a}$. The expansion tensors $\Delta^{(n)}_{a_1\cdots a_n}$ are determined by collecting terms according to their order in σ and setting to zero.

From Eqs. (B20) and (B25d), we have the series expansion

$$\Box\sigma = 4 - \frac{2}{3}\sigma p^p p^q R_{pq} + \frac{\sqrt{2}}{6}\sigma^{3/2} R_{pq;r} p^p p^q p^r$$
$$- \frac{1}{45}\sigma^2 p^p p^q p^r p^s (3R_{pq;rs} + 4R_{ptqu} R_r{}^u{}_s{}^t)$$
$$+ \frac{\sqrt{2}}{90}\sigma^{5/2} p^p p^q p^r p^s p^t (R_{pq;rst} + 6R_{pvqu;r} R_s{}^v{}_t{}^u)$$
$$+ \frac{1}{1890}\sigma^3 p^p p^q p^r p^s p^t p^u (219R_{pvqw;r} R_s{}^v{}_t{}^w{}_{;u}$$
$$- 6R_{pq;rstu} + 24R_{pvqw} R_r{}^w{}_{sx} R_t{}^v{}_u{}^x + 228R_{pwqv;rs} R_t{}^w{}_u{}^v$$
$$+ 8R_{pvqw} R_r{}^w{}_{sx} R_t{}^x{}_u{}^v. \tag{C7}$$

We have made the split into σ and p^a so we can readily carry out the needed multiplication in the series (C6), once we put it into the same form. The last piece we need is $\Delta^{1/2}_{;a}$. This is obtained by differentiating Eq. (C6) and then substituting Eq. (B20) for the $\sigma_{;ab}$ terms that arise. Once this is done, we have all the terms for the series expansion of Eq. (C3). The order σ term is

$$\frac{2\sigma p^p p^q}{3}[12\Delta^{(2)}{}_{pq} - R_{pq}] = 0 \Rightarrow \Delta^{(2)}{}_{ab} = \frac{R_{ab}}{12}. \tag{C8}$$

The order $\sigma^{3/2}$ term starts off as

$$\frac{\sigma^{3/2} p^p p^q p^r}{3\sqrt{2}}[24\Delta^{(2)}{}_{pq;r} + R_{pq;r} + 72\Delta^{(3)}{}_{pqr}]$$
$$= 0 \Rightarrow \Delta^{(3)}{}_{abc} \doteq -\frac{1}{72}[24\Delta^{(2)}{}_{ab;c} + R_{ab;c}]. \tag{C9}$$

Using Eq. (C8) shows

$$\Delta^{(3)}{}_{abc} \doteq -\frac{R_{ab;c}}{24}. \tag{C10}$$

For the order σ^2 term, we find

$$\Delta^{(4)}{}_{abcd} \doteq \frac{1}{1440}[-360\Delta^{(3)}{}_{abc;d} + 3R_{ab;cd} + 60\Delta^{(2)}{}_{ab} R_{cd}$$
$$+ 120\Delta^{(2)}{}_{pa} R_{bcd}{}^p + 120\Delta^{(2)}{}_{ap} R_b{}^p{}_{cd}$$
$$+ 4R_{paqb} R_c{}^q{}_d{}^p]. \tag{C11}$$

The fourth and fifth terms above vanish since we only need equality up to symmetrization. Using Eqs. (C8) and (C10), the final form for this term becomes

$$\Delta^{(4)}{}_{abcd} \doteq \frac{1}{1440}(18R_{ab;cd} + 5R_{ab}R_{cd} + 4R_{paqb} R_c{}^q{}_d{}^p). \tag{C12a}$$

The last two coefficients are determined in exactly the same manner. The results are

$$\Delta^{(5)}{}_{abcde} \doteq -\frac{1}{1440}(4R_{ab;cde} + 5R_{ab;c}R_{de} + 4R_{paqb;c} R_d{}^q{}_e{}^p) \tag{C12b}$$

$$\Delta^{(6)}{}_{abcdef} \doteq \frac{1}{362880}(315R_{ab;c}R_{de;f} + 180R_{ab;cdef}$$
$$+ 378R_{ab;cd}R_{ef} + 35R_{ab}R_{cd}R_{ef}$$
$$+ 84R_{ab}R_{pcqd}R_e{}^q{}_f{}^p - 270R_{paqb;c}R_d{}^q{}_e{}^p{}_{;f}$$
$$- 288R_{paqb;cd}R_e{}^q{}_f{}^p + 64R_{paqb}R_{rdc}{}^q R_e{}^r{}_f{}^p). \tag{C12c}$$

APPENDIX D: SERIES EXPANSION FOR $\Delta\tau$

In this section we determine the expansion

$$\Delta\tau^2 = \sigma^{;p}\sigma^{;q}\delta\tau^{(2)}{}_{pq} + \sigma^{;p}\sigma^{;q}\sigma^{;r}\delta\tau^{(3)}{}_{pqr}$$
$$+ \sigma^{;p}\sigma^{;q}\sigma^{;r}\sigma^{;s}\delta\tau^{(4)}{}_{pqrs}. \tag{D1}$$

Since this is the expansion of a symmetric function, the expansion tensors are related to the coincident limit of the covariant derivatives of $\Delta\tau^2$ via Eqs. (B17e) and (B18d). We also use $[\Delta\tau] = 0$, justifying that the series expansion starts at order two in σ^a. The expansion tensors are

$$\delta\tau^{(2)}{}_{ab} \doteq [\Delta\tau_{;a}][\Delta\tau_{;b}], \tag{D2a}$$

$$\delta\tau^{(3)}{}_{abc} \doteq -([\Delta\tau_{;a}]_{;b}[\Delta\tau_{;c}]), \tag{D2b}$$

$$\delta\tau^{(4)}{}_{abcd} \doteq \frac{[\Delta\tau_{;ab}][\Delta\tau_{;cd}]}{4} + \frac{[\Delta\tau_{;a}][\Delta\tau_{;bcd}]}{3}. \tag{D2c}$$

To evaluate the covariant derivatives, we start with $\Delta\tau_{;a} = \Delta\tau_{,a} = \delta_a{}^\tau$. Also, $\Delta\tau_{,ab} = 0$ and $[\Delta\tau_{;a}] = \Delta\tau_{;a} \Rightarrow [\Delta\tau_{;a}]_{;b} = \Delta\tau_{;ab}$.

Turning to the computation of two covariant derivatives,

$$\Delta\tau_{;ab} = \Delta\tau_{;a,b} - \Gamma^p{}_{ab}\Delta\tau_{;p}$$
$$= \Delta\tau_{;a,b} - \Gamma^p{}_{ab}\Delta\tau_{;p}$$
$$= -\Gamma^p{}_{ab}\delta_p{}^\tau$$
$$= -\Gamma^\tau{}_{ab}, \tag{D3}$$

and three covariant derivatives,

$$\Delta\tau_{;abc} = -\Gamma^\tau{}_{ab;c}. \tag{D4}$$

Using these results, the expansion tensors (D2c) become

$$\delta\tau^{(2)}{}_{ab} = \delta_a{}^\tau\delta_b{}^\tau, \tag{D5a}$$

$$\delta\tau^{(3)}{}_{abc} = \Gamma^\tau{}_{ab}\delta_c{}^\tau, \tag{D5b}$$

$$\delta\tau^{(4)}{}_{abcd} = \frac{1}{4}\Gamma^\tau{}_{ab}\Gamma^\tau{}_{cd} - \frac{1}{3}\Gamma^\tau{}_{ab;c}\delta_d{}^\tau. \tag{D5c}$$

APPENDIX E: SERIES EXPANSION OF HADAMARD FORM

For the regularization of the coincident limit of the noise kernel, we need the Hadamard form,

$$S(x,y)=\frac{1}{(4\pi)^2}\left[\frac{2\Delta^{1/2}}{\sigma}+(v_0+\sigma v_1+\sigma^2 v_2)\ln\sigma + (\sigma w_1+\sigma^2 w_2)\right], \tag{E1}$$

to fourth order in σ^a. We review the standard techniques for finding these expansions and present the results we use. The functions $v_n(x,y)$ and $w_n(x,y)$, $n\geqslant 1$ are determined by demanding

$$(\Box - R/6)S(x,y)=0. \tag{E2}$$

The arbitrary function $w_0(x,y)$ is assumed to vanish. Working to fourth order, we proceed in the now familiar pattern of expanding each of the functions in a series expansion and then solve for the expansion tensors by putting the expansion in the equations derived from Eq. (E2). Using the differential operators

$$H_n=\sigma^{;p}\nabla_p+\left\{\left[n-1+\frac{1}{2}(\Box\sigma)\right]\right\}, \tag{E3a}$$

$$K=\Box-\frac{R}{6} \tag{E3b}$$

we need to solve

$$v_0(x,y)=v_0^{(0)}+\sigma^p v_{0p}^{(1)}+\sigma^p\sigma^q v_{0pq}^{(2)}+\sigma^p\sigma^q\sigma^r v_{0pqr}^{(3)} +\sigma^p\sigma^q\sigma^r\sigma^s v_{0pqrs}^{(4)}, \tag{E4a}$$

$$H_0 v_0=-K\Delta^{1/2}, \tag{E4b}$$

$$v_1(x,y)\approx v_1^{(0)}+\sigma^p v_{1p}^{(1)}+\sigma^p\sigma^q v_{1pq}^{(2)}, \tag{E5a}$$

$$2H_1 v_1=-Kv0, \tag{E5b}$$

$$v_2(x,y)\approx v_2^{(0)}, \tag{E6a}$$

$$4H_2 v_2=-Kv1, \tag{E6b}$$

along with

$$w_1(x,y)\approx w_1^{(0)}+\sigma^p w_{1p}^{(1)}+\sigma^p\sigma^q w_{1pq}^{(2)}, \tag{E7a}$$

$$2H_1 w_1+2H_2 v1=0, \tag{E7b}$$

$$w_2(x,y)\approx w_2^{(0)}, \tag{E8a}$$

$$4H_2 w_2+2H_4 v_2=-Kw1. \tag{E8b}$$

Proceeding as in Appendix C, we find

$$v_0^{(0)}=0, \tag{E9a}$$

$$v_{0a}^{(1)}=0, \tag{E9b}$$

$$v_{0ab}^{(2)}=(R_{;ab}-3R_{ab;p}{}^{p}+4R_{pa}R_{b}{}^{p}-2R_{pq}R_{a}{}^{q}{}_{b}{}^{p}+2R_{pqra}R_{b}{}^{rpq})/360, \tag{E9c}$$

$$\begin{aligned}v_{0abc}^{(3)}=(&5R_{;abc}-14R_{pa;bc}{}^{p}-7R_{ab;pc}{}^{p}+5R_{ab;p}{}^{p}{}_{c}+8R_{ab;cp}{}^{p}+10R_{pa;b}R_{c}{}^{p}+15R_{ab;p}R_{c}{}^{p}-10R_{paqb;c}R^{pq}-24R_{pq;a}R_{b}{}^{q}{}_{c}{}^{p}\\&+4R_{pa;q}R_{b}{}^{q}{}_{c}{}^{p}-8R_{paqr;b}R_{c}{}^{pqr}+2R_{paqb;r}R_{c}{}^{pqr}-2R_{paqr;b}R_{c}{}^{qpr}+2R_{paqb;r}R_{c}{}^{qpr}+2R_{paqr;b}R_{c}{}^{rpq})/1440,\end{aligned} \tag{E9d}$$

$$\begin{aligned}v_{0abcd}^{(4)}=(&-3150R_{pa;b}R_{cd}{}^{;p}-2205R_{ab;p}R_{cd}{}^{;p}-720R_{pa;b}R_{c}{}^{p}{}_{;d}+11160R_{pq;a}R_{b}{}^{p}{}_{c}{}^{q}{}_{;d}-3600R_{pa;q}R_{b}{}^{p}{}_{c}{}^{q}{}_{;d}+270R_{paqb;r}R_{c}{}^{p}{}_{d}{}^{q;r}\\&+2556R_{pqra;b}R_{c}{}^{p}{}_{d}{}^{r;q}+36R_{pqra;b}R_{c}{}^{pqr}{}_{;d}-522R_{paqb;r}R_{c}{}^{pqr}{}_{;d}-522R_{paqb;r}R_{c}{}^{qpr}{}_{;d}+468R_{pqra;b}R_{c}{}^{rpq}{}_{;d}-1170R_{;abcd}\\&+1656R_{pa;bcd}{}^{p}+1404R_{pa;bc}{}^{p}{}_{d}+828R_{ab;pcd}{}^{p}+702R_{ab;pc}{}^{p}{}_{d}-810R_{ab;p}{}^{p}{}_{cd}+828R_{ab;cpd}{}^{p}-1188R_{ab;cp}{}^{p}{}_{d}\\&-1440R_{ab;cdp}{}^{p}+210R_{;ab}R_{cd}-630R_{ab;p}{}^{p}R_{cd}-3996R_{pa;bc}R_{d}{}^{p}-4914R_{ab;pc}R_{d}{}^{p}+1008R_{ab;cp}R_{d}{}^{p}+840R_{pa}R_{bc}R_{d}{}^{p}\\&+2916R_{paqb;cd}R^{pq}-336R_{;pa}R_{bcd}{}^{p}+1008R_{pa;q}{}^{q}R_{bcd}{}^{p}+1344R_{pq}R_{a}{}^{q}R_{bcd}{}^{p}-336R_{pa}R_{qrb}{}^{p}R_{cd}{}^{qr}+10152R_{pq;ab}R_{c}{}^{p}{}_{d}{}^{q}\\&+2088R_{pa;qb}R_{c}{}^{p}{}_{d}{}^{q}-4320R_{pa;bq}R_{c}{}^{p}{}_{d}{}^{q}-1728R_{ab;pq}R_{c}{}^{p}{}_{d}{}^{q}+672R_{pa}R_{qb}R_{c}{}^{p}{}_{d}{}^{q}-420R_{pq}R_{ab}R_{c}{}^{p}{}_{d}{}^{q}\\&+4728R_{pa}R_{qbr}{}^{p}R_{c}{}^{q}{}_{d}{}^{r}+1536R_{pq}R_{rab}{}^{p}R_{c}{}^{q}{}_{d}{}^{r}+64R_{pqra}R_{sb}{}^{pr}R_{c}{}^{q}{}_{d}{}^{s}+64R_{pqra}R_{s}{}^{p}{}_{b}{}^{r}R_{c}{}^{q}{}_{d}{}^{s}+3168R_{pqra}R_{s}{}^{r}{}_{b}{}^{p}R_{c}{}^{q}{}_{d}{}^{s}\\&-1392R_{pqra}R_{sb}{}^{pq}R_{c}{}^{r}{}_{d}{}^{s}+72R_{pqra;bc}R_{d}{}^{pqr}-540R_{paqb;rc}R_{d}{}^{pqr}-1296R_{paqb;cr}R_{d}{}^{pqr}-672R_{pq}R_{rabc}R_{d}{}^{pqr}\\&+1344R_{pqra}R_{sbc}{}^{q}R_{d}{}^{prs}-540R_{paqb;rc}R_{d}{}^{qpr}-1296R_{paqb;cr}R_{d}{}^{qpr}-252R_{pa}R_{qbrc}R_{d}{}^{qpr}+432R_{pqra;bc}R_{d}{}^{rpq}\end{aligned}$$

$$+420R_{ab}R_{pqrc}R_d{}^{rpq}-252R_{pa}R_{qbrc}R_d{}^{rpq}+1344R_{pqra}R_{sbc}{}^{q}R_d{}^{rps}-128R_{pqra}R_{sbc}{}^{p}R_d{}^{rqs}+96R_{pqra}R_{sbc}{}^{r}R_d{}^{spq}$$
$$-128R_{pqra}R_{sbc}{}^{p}R_d{}^{sqr}-672R_{pqra}R_{sbcd}R^{pqrs}-1200R_{paqb}R_{rcsd}R^{prqs}-624R_{paqb}R_{rcsd}R^{psqr}/907200, \quad \text{(E9e)}$$

$$v_1^{(0)}=(R_{;p}{}^{p}-R_{pq}R^{pq}+R_{pqrs}R^{pqrs})/360, \quad \text{(E10a)}$$

$$v_{1a}^{(1)}=(-6R_{;pa}{}^{p}-14R_{;pa}{}^{p}+28R_{pa;q}{}^{qp}-22R_{;p}R_a{}^{p}+40R_{pq;a}R^{pq}-56R_{pa;q}R^{pq}-37R_{pq;r}R_a{}^{pqr}-19R_{pq;r}R_a{}^{qpr}$$
$$-17R_{pqrs;a}R^{pqrs}+2R_{pqrs;a}R^{psqr}+12R_{paqr;s}R^{psqr})/4320, \quad \text{(E10b)}$$

$$v_{1ab}^{(2)}=(210R_{;p}R_{ab}{}^{;p}+380R_{;p}R_a{}^{p}{}_{;b}-360R_{pa;q}R_b{}^{p;q}-2280R_{pa;q}R_b{}^{q;p}-800R_{pq;a}R^{pq}{}_{;b}-200R_{pa;q}R^{pq}{}_{;b}-1488R_{;pab}{}^{p}$$
$$+528R_{;pa}{}^{p}{}_{b}+820R_{;p}{}^{p}{}_{ab}+180R_{;abp}{}^{p}+960R_{pq;ab}{}^{pq}+960R_{pa;qb}{}^{pq}+960R_{pa;qb}{}^{qp}-1056R_{pa;q}{}^{q}{}_{b}{}^{p}-944R_{pa;q}{}^{qp}{}_{b}$$
$$+960R_{pa;bq}{}^{pq}-1560R_{pa;bq}{}^{qp}+480R_{ab;pq}{}^{pq}-780R_{ab;pq}{}^{qp}+480R_{ab;p}{}^{p}{}_{q}{}^{q}+140R_{;p}{}^{p}R_{ab}-152R_{;pa}R_b{}^{p}-264R_{pa;q}{}^{q}R_b{}^{p}$$
$$-1232R_{pq;ab}R^{pq}+1768R_{pa;qb}R^{pq}-1056R_{pa;bq}R^{pq}-132R_{ab;pq}R^{pq}-3680R_{pa}R_{qb}R^{pq}-140R_{pq}R_{ab}R^{pq}$$
$$+1320R_{pq;r}R_a{}^{p}{}_{b}{}^{q;r}+360R_{pq;r}R_a{}^{p}{}_{b}{}^{r;q}+5560R_{pq;r}R_a{}^{pqr}{}_{;b}-516R_{pqra;s}R_b{}^{pqr;s}+180R_{pqra;s}R_b{}^{prs;q}+12R_{pqra;s}R_b{}^{rpq;s}$$
$$+180R_{pqra;s}R_b{}^{rps;q}+705R_{pqrs;a}R_{;b}{}^{pqrs}-1716R_{pqra;s}R^{pqrs}{}_{;b}-1008R_{pqra;s}R^{prqs}{}_{;b}+544R_{;pq}R_a{}^{p}{}_{b}{}^{q}-72R_{pq;r}{}^{r}R_a{}^{p}{}_{b}{}^{q}$$
$$+1256R_{pq}R_r{}^{p}R_a{}^{q}{}_{b}{}^{r}+6832R_{pq;ra}R_b{}^{pqr}-672R_{pq;ar}R_b{}^{pqr}+1728R_{pa;qr}R_b{}^{pqr}-4712R_{pq}R_{ra}R_b{}^{pqr}+720R_{pa;qr}R_b{}^{qpr}$$
$$-246R_{pq}R_{rsa}{}^{p}R_b{}^{qrs}-784R_{pq}R_{ras}{}^{p}R_b{}^{qrs}-276R_{pq}R_{rsa}{}^{p}R_b{}^{rqs}-464R_{pq}R_{ras}{}^{p}R_b{}^{rqs}+1472R_{pq}R_{ras}{}^{p}R_b{}^{sqr}$$
$$+664R_{pqrs;ab}R^{pqrs}-576R_{pqra;sb}R^{pqrs}-1416R_{pqra;bs}R^{pqrs}+140R_{ab}R_{pqrs}R^{pqrs}-1020R_{pqra}R_{stb}{}^{r}R^{pqst}$$
$$+64R_{pqrs;ab}R^{prqs}-96R_{pqra;sb}R^{prqs}-432R_{pqra;bs}R^{prqs}-192R_{paqb;rs}R^{prqs}-2576R_{pq}R_{rasb}R^{prqs}-432R_{pqra}R_{stb}{}^{q}R^{prst}$$
$$+1624R_{pa}R_{qrsb}R^{psqr}+240R_{pqra}R_{stb}{}^{r}R^{psqt}+240R_{pqra}R_{stb}{}^{q}R^{psrt}+992R_{pqrs}R_{tab}{}^{q}R^{ptrs}-16R_{pqra}R_{stb}{}^{p}R^{qrst}$$
$$-2068R_{pqra}R_{sbt}{}^{p}R^{qsrt}-1972R_{pqra}R_{stb}{}^{p}R^{qtrs}-2988R_{pqra}R_{sbt}{}^{p}R^{qtrs}/604800, \quad \text{(E10c)}$$

$$v_2^{(0)}=(139R_{;p}R^{;p}-354R_{pq;r}R^{pq;r}+732R_{pq;r}R^{pr;q}+114R_{;p}{}^{p}{}_{q}{}^{q}-288R_{pq;r}{}^{rqp}+508R_{;pq}R^{pq}-84R_{pq;r}{}^{r}R^{pq}$$
$$+484R_{pq}R_r{}^{p}R^{qr})/302400+(405R_{pqrs;t}R^{pqrs;t}+810R_{pqrs;t}R^{pqrt;s}+288R_{pq;rs}R^{prqs}-5520R_{pq}R_{rs}R^{prqs}$$
$$-2352R_{pq}R_{rst}{}^{p}R^{qtrs}-3520R_{pqrs}R_t{}^{p}{}_{u}{}^{r}R^{qtsu}-640R_{pqrs}R_{tu}{}^{pq}R^{rstu})/3628800, \quad \text{(E11)}$$

$$w_1^{(0)}=-(R_{;p}{}^{p}-R_{pq}R^{pq}+R_{pqrs}R^{pqrs})/240, \quad \text{(E12a)}$$

$$w_{1a}^{(1)}=(12R_{;pa}{}^{p}+31R_{;p}{}^{p}{}_{a}-56R_{pa;q}{}^{qp}+44R_{;p}R_a{}^{p}-86R_{pq;a}R^{pq}+112R_{pa;q}R^{pq}+74R_{pq;r}R_a{}^{pqr}+38R_{pq;r}R_a{}^{qpr}$$
$$+40R_{pqrs;a}R^{pqrs}-4R_{pqrs;a}R^{psqr}-24R_{paqr;s}R^{psqr})/6480, \quad \text{(E12b)}$$

$$w_{1ab}^{(2)}=(-630R_{;p}R_{ab}{}^{;p}-1756R_{;p}R_a{}^{p}{}_{;b}+1080R_{pa;q}R_b{}^{p;q}+6840R_{pa;q}R_b{}^{q;p}+3856R_{pq;a}R^{pq}{}_{;b}-968R_{pa;q}R^{pq}{}_{;b}+4464R_{;pab}{}^{p}$$
$$-1752R_{;pa}{}^{p}{}_{b}-3020R_{;p}{}^{p}{}_{ab}-540R_{;abp}{}^{p}-2880R_{pq;ab}{}^{pq}-2880R_{pa;qb}{}^{pq}-2880R_{pa;qb}{}^{qp}+3168R_{pa;q}{}^{q}{}_{b}{}^{p}+3616R_{pa;qb}{}^{qp}$$
$$-2880R_{pa;bq}{}^{pq}+4680R_{pa;bq}{}^{qp}-1440R_{ab;pq}{}^{pq}+2340R_{ab;pq}{}^{qp}-1440R_{ab;p}{}^{p}{}_{q}{}^{q}-504R_{;p}{}^{p}R_{ab}-160R_{;pa}R_b{}^{p}$$
$$+792R_{pa;q}{}^{q}R_b{}^{p}+5152R_{pq;ab}R^{pq}-6872R_{pa;qb}R^{pq}+3168R_{pa;bq}R^{pq}+396R_{ab;pq}R^{pq}+11040R_{pa}R_{qb}R^{pq}$$
$$+504R_{pq}R_{ab}R^{pq}-3960R_{pq;r}R_a{}^{p}{}_{b}{}^{q;r}-1080R_{pq;r}R_a{}^{p}{}_{b}{}^{r;q}-18248R_{pq;r}R_a{}^{pqr}{}_{;b}+1548R_{pqra;s}R_b{}^{pqr;s}-540R_{pqra;s}R_b{}^{prs;q}$$
$$-36R_{pqra;s}R_b{}^{rpq;s}-540R_{pqra;s}R_b{}^{rps;q}-2955R_{pqrs;a}R^{pqrs}{}_{;b}+5484R_{pqra;s}R^{pqrs}{}_{;b}+3024R_{pqra;s}R^{prqs}{}_{;b}$$
$$-1632R_{;pq}R_a{}^{p}{}_{b}{}^{q}+216R_{pq;r}{}^{r}R_a{}^{p}{}_{b}{}^{q}-3768R_{pq}R_r{}^{p}R_a{}^{q}{}_{b}{}^{r}-22064R_{pq;ra}R_b{}^{pqr}+2016R_{pq;ar}R_b{}^{pqr}-5184R_{pa;qr}R_b{}^{pqr}$$
$$+14136R_{pq}R_{ra}R_b{}^{pqr}-2160R_{pa;qr}R_b{}^{qpr}+738R_{pq}R_{rsa}{}^{p}R_b{}^{qrs}+2352R_{pq}R_{ras}{}^{p}R_b{}^{qrs}+828R_{pq}R_{rsa}{}^{p}R_b{}^{rqs}$$
$$+1392R_{pq}R_{ras}{}^{p}R_b{}^{rqs}-4416R_{pq}R_{ras}{}^{p}R_b{}^{sqr}-2804R_{pqrs;ab}R^{pqrs}+2064R_{pqra;sb}R^{pqrs}+4248R_{pqra;bs}R^{pqrs}$$
$$-504R_{ab}R_{pqrs}R^{pqrs}+3060R_{pqra}R_{stb}{}^{r}R^{pqst}-248R_{pqrs;ab}R^{prqs}+288R_{pqra;sb}R^{prqs}+1296R_{pqra;bs}R^{prqs}$$

$$+576R_{paqb;rs}R^{prqs}+7728R_{pq}R_{rasb}R^{prqs}+1296R_{pqra}R_{stb}{}^{q}R^{prst}-4872R_{pa}R_{qrsb}R^{psqr}-720R_{pqra}R_{stb}{}^{r}R^{psqt}$$
$$-720R_{pqra}R_{stb}{}^{q}R^{psrt}-2976R_{pqrs}R_{tab}{}^{q}R^{ptrs}+48R_{pqra}R_{stb}{}^{p}R^{qrst}+6204R_{pqra}R_{sbt}{}^{p}R^{qsrt}+5916R_{pqra}R_{stb}{}^{p}R^{qtrs}$$
$$+8964R_{pqra}R_{sbt}{}^{p}R^{qtrs})/1451520, \tag{E12c}$$

$$w_2^{(0)}=-(139R_{;p}R^{;p}-354R_{pq;r}R^{pq;r}+732R_{pq;r}R^{pr;q}+114R_{;p}{}^{p}{}_{q}{}^{q}-288R_{pq;r}{}^{rqp}+508R_{;pq}R^{pq}-84R_{pq;r}{}^{r}R^{pq}$$
$$+484R_{pq}R_{r}{}^{p}R^{qr})/145152-(405R_{pqrs;t}R^{pqrs;t}+810R_{pqrs;t}R^{pqrt;s}+288R_{pq;rs}R^{prqs}-5520R_{pq}R_{rs}R^{prqs}$$
$$-2352R_{pq}R_{rst}{}^{p}R^{qtrs}-3520R_{pqrs}R_{t}{}^{p}{}_{u}{}^{r}R^{qtsu}-640R_{pqrs}R_{tu}{}^{pq}R^{rstu})/1741824. \tag{E13}$$

APPENDIX F: CONFORMAL TRANSFORMATION

Since we compute the Green function in the optical metric conformally related to the physical metric we are interested in, we need to know the way in which the geometric objects we use conformally transform. We denote objects in the optical metric with an overbar and objects in the covariant derivative commensurate with the optical metric by a vertical stroke. The two metrics are related via

$$\overline{g}_{ab}=e^{-2\omega}g_{ab}\,. \tag{F1}$$

We develop the series expansions of the world function $\overline{\sigma}$ to eighth order and $\overline{\Delta}^{1/2}$ to sixth order. We also need the function

$$\Sigma(x,y)=e^{\omega(x)+\omega(y)}\overline{\sigma}-\sigma, \tag{F2}$$

whose series expansion readily follows once the series for $\overline{\sigma}$ is determined. The most straightforward method for determining these series expansions is to start by considering the conformal transformation that the differential equations each must satisfy.

Considering $\overline{\sigma}$ first, it satisfies, in terms of the optical metric, the equation

$$\overline{\sigma}=\frac{1}{2}\overline{\sigma}_{|p}\overline{\sigma}^{|p}=\frac{1}{2}\overline{g}^{pq}\overline{\sigma}_{|p}\overline{\sigma}_{|q}\,. \tag{F3}$$

Using $\overline{\sigma}_{|a}=\overline{\sigma}_{,a}=\overline{\sigma}_{;a}$, this equation transforms to

$$\overline{\sigma}=\frac{e^{2\omega}}{2}g^{pq}\overline{\sigma}_{;p}\overline{\sigma}_{;q}=\frac{e^{2\omega}}{2}\overline{\sigma}_{;p}\overline{\sigma}^{;p}, \tag{F4}$$

and we now have an equation purely in terms of the physical metric. We still have

$$[\overline{\sigma}]=0, \quad [\overline{\sigma}_{;a}]=0. \tag{F5}$$

and we assume the series expansion

$$\overline{\sigma}=\Upsilon^{(2)}_{pq}\sigma^p\sigma^q+\Upsilon^{(3)}_{pqr}\sigma^p\sigma^q\sigma^r+\Upsilon^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s$$
$$+\Upsilon^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t+\Upsilon^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u$$
$$+\Upsilon^{(7)}_{pqrstuv}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v$$
$$+\Upsilon^{(8)}_{pqrstuvw}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v\sigma^w. \tag{F6}$$

To determine the expansion tensors, we calculate as before for $\Delta^{1/2}$, substitute the expansion into the differential Eq. (F4), and collect terms by their order in σ^a.

The order σ term must satisfy

$$2\Upsilon^{(2)}{}_{ab}-e^{2\omega}[\Upsilon^{(2)}{}_{ap}\Upsilon^{(2)}{}_{b}{}^{p}+\Upsilon^{(2)}{}_{pa}\Upsilon^{(2)p}{}_{b}+2\Upsilon^{(2)}{}_{ap}\Upsilon^{(2)p}{}_{b}]$$
$$=0. \tag{F7}$$

This has the solution

$$\Upsilon^{(2)}{}_{ab}=\frac{g_{ab}}{2e^{2\omega}}, \tag{F8a}$$

which can be seen via substitution. Since $\overline{\sigma}$ is a symmetric function, we use the results of Appendix B, which give the odd order expansion coefficients in terms of the even order one. Thus the third order coefficient is

$$\Upsilon^{(3)}{}_{abc}=-\frac{\Upsilon^{(2)}{}_{ab;c}}{2}=\frac{\omega_{;c}g_{ab}}{2e^{2\omega}}. \tag{F8b}$$

With this, we have set up the recursion. One then proceeds with the computation of the series expansion Van Vleck–Morette determinant as before. The results for the rest of the expansion tensors for $\overline{\sigma}$ are

$$e^{2\omega}\Upsilon^{(4)}{}_{abcd}\doteq\frac{g_{cd}}{6}(2\omega_{;a}\omega_{;b}-\omega_{;ab})-\frac{g_{ab}g_{cd}}{24}\omega_{;p}\omega^{;p}, \tag{F8c}$$

$$e^{2\omega}\Upsilon^{(5)}{}_{abcde} \doteq \frac{g_{ab}}{24}(4\omega_{;c}\omega_{;d}\omega_{;e} - 6\omega_{;c}\omega_{;de} + \omega_{;cde}) - \frac{g_{ab}g_{cd}}{24}\omega_{;p}(\omega_{;e}\omega^{;p} - \omega^{p}_{;e}). \tag{F8d}$$

Here $\doteq$ denotes equality upon symmetrization. The expressions for $e^{2\omega}\Upsilon^{(6)}{}_{abcdef}$, $e^{2\omega}\Upsilon^{(7)}{}_{abcdefg}$, and $e^{2\omega}\Upsilon^{(8)}{}_{abcdefgh}$ can be found in Ref. [48].

By using the expansion

$$\begin{aligned}
e^{\omega+\omega'} = e^{2\omega}\Bigg[& 1 - \sqrt{2}\sqrt{\sigma}p^p\omega_{;p} + \sigma p^p p^q(\omega_{;p}\omega_{;q} + \omega_{;pq}) - \frac{\sqrt{2}}{3}\sigma^{3/2}p^p p^q p^r(\omega_{;p}\omega_{;q}\omega_{;r} + 3\omega_{;p}\omega_{;qr} + \omega_{;pqr}) \\
& + \frac{1}{6}\sigma^2 p^p p^q p^r p^s(\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;s} + 6\omega_{;p}\omega_{;q}\omega_{;rs} + 3\omega_{;pq}\omega_{;rs} + 4\omega_{;p}\omega_{;qrs} + \omega_{;pqrs}) \\
& - \frac{1}{15\sqrt{2}}\sigma^{5/2}p^p p^q p^r p^s p^t(\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;s}\omega_{;t} + 10\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;st} + 15\omega_{;p}\omega_{;qr}\omega_{;st} + 10\omega_{;p}\omega_{;q}\omega_{;rst} + 10\omega_{;pq}\omega_{;rst} \\
& + 5\omega_{;p}\omega_{;qrst} + \omega_{;pqrst}) + \frac{1}{90}\sigma^3 p^p p^q p^r p^s p^t p^u(\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;s}\omega_{;t}\omega_{;u} + 15\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;s}\omega_{;tu} + 45\omega_{;p}\omega_{;q}\omega_{;rs}\omega_{;tu} \\
& + 15\omega_{;pq}\omega_{;rs}\omega_{;tu} + 20\omega_{;p}\omega_{;q}\omega_{;r}\omega_{;stu} + 60\omega_{;p}\omega_{;qr}\omega_{;stu} + 10\omega_{;pqr}\omega_{;stu} + 15\omega_{;p}\omega_{;q}\omega_{;rstu} + 15\omega_{;pq}\omega_{;rstu} \\
& + 6\omega_{;p}\omega_{;qrstu} + \omega_{;pqrstu})\Bigg] + O(\sigma^{7/2})
\end{aligned} \tag{F9}$$

we can get the expansion of Σ. Multiplying together the above series and Eq. (F6), and subtracting σ, we determine the expansion

$$\begin{aligned}
\Sigma = {} & \Sigma^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s + \Sigma^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t \\
& + \Sigma^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u + \Sigma^{(7)}_{pqrstuv}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v \\
& + \Sigma^{(8)}_{pqrstuvw}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u\sigma^v\sigma^w,
\end{aligned} \tag{F10}$$

where the expansion tensors are

$$\Sigma^{(4)}_{abcd} \doteq \frac{g_{ab}}{3}(\omega_{;c}\omega_{;d} + \omega_{;cd}) - \frac{g_{ab}g_{cd}}{6}\omega_{;p}\omega^{;p}, \tag{F11a}$$

$$\Sigma^{(5)}_{abcde} \doteq -\frac{g_{ab}}{3\sqrt{2}}(2\omega_{;c}\omega_{;de} + \omega_{;cde}) + \frac{g_{ab}g_{cd}}{3\sqrt{2}}\omega_{;p}\omega_{;e}{}^{p}, \tag{F11b}$$

$$\begin{aligned}
\Sigma^{(6)}_{abcdef} \doteq {} & \frac{g_{ab}}{30}(\omega_{;c}\omega_{;d}\omega_{;e}\omega_{;f} + 2\omega_{;c}\omega_{;d}\omega_{;ef} + 7\omega_{;cd}\omega_{;ef} \\
& + 6\omega_{;c}\omega_{;def} + 3\omega_{;cdef}) - \frac{g_{ab}g_{cd}}{90}(3\omega_{;r}\omega_{;e}\omega_{;f}\omega^{;r} \\
& + 7\omega_{;p}\omega^{;p}\omega_{;ef} + 9\omega_{;p}\omega_{;ef}{}^{p} + 12\omega_{;p}\omega_{;q}R_{e}{}^{q}{}_{f}{}^{p}) \\
& + \frac{1}{720}8g_{ab}g_{cd}g_{ef}\omega_{;p}\omega_{;q}(\omega^{;p}\omega^{;q} - 3\omega^{;pq}).
\end{aligned} \tag{F11c}$$

The expressions for $\Sigma^{(7)}_{abcdefg}$ and $\Sigma^{(8)}_{abcdefgh}$ can be found in Ref. [48].

Turning to the last expansion we need, we recall that the Van Vleck–Morette determinant on the optical metric, $\bar{\Delta}^{1/2}$, satisfies the differential equation

$$\bar{\Delta}^{1/2}(4 - \bar{\Box}\bar{\sigma}) - 2\bar{\Delta}^{1/2}{}_{|p}\bar{\sigma}^{|p} = 0. \tag{F12}$$

Using the conformal transformation property

$$\bar{\Box}\bar{\sigma} = e^{2\omega}(\Box\bar{\sigma} - 2\omega^{;p}\bar{\sigma}_{;p}) \tag{F13}$$

we determine that $\bar{\Delta}^{1/2}$ satisfies the equation

$$\bar{\Delta}^{1/2}[4 - e^{2\omega}(\Box\bar{\sigma} - 2\omega^{;p}\bar{\sigma}_{;p})] - 2e^{2\omega}\bar{\Delta}^{1/2}{}_{;p}\bar{\sigma}^{;p} = 0 \tag{F14}$$

on the physical metric. Since we have the expansion for $\bar{\sigma}$, we need only assume the expansion

$$\begin{aligned}
\bar{\Delta}^{1/2} = {} & 1 + \bar{\Delta}^{(2)}_{pq}\sigma^p\sigma^q + \bar{\Delta}^{(3)}_{pqr}\sigma^p\sigma^q\sigma^r + \bar{\Delta}^{(4)}_{pqrs}\sigma^p\sigma^q\sigma^r\sigma^s \\
& + \bar{\Delta}^{(5)}_{pqrst}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t + \bar{\Delta}^{(6)}_{pqrstu}\sigma^p\sigma^q\sigma^r\sigma^s\sigma^t\sigma^u,
\end{aligned} \tag{F15}$$

substitute this and Eq. (F6) into (F14), and solve for the expansion tensors. Following the now well defined method outlined above, they are found to be

$$\overline{\Delta}^{(2)}_{ab} \doteq \Delta^{(2)}_{ab} + \frac{1}{6}(\omega_{;a}\omega_{;b} + \omega_{;ab}) - \frac{g_{ab}}{12}(2\omega_{;p}\omega^{;p} - \omega_{;p}{}^{p}, \tag{F16a}$$

$$\overline{\Delta}^{(3)}_{abc} \doteq \Delta^{(3)}_{abc} - \frac{1}{12}(2\omega_{;a}\omega_{;bc} + \omega_{;abc}) + \frac{g_{ab}}{24}(4\omega_{;p}\omega_{;c}{}^{p} - \omega_{;pc}{}^{p}) \tag{F16b}$$

$$\begin{aligned}\overline{\Delta}^{(4)}_{abcd} \doteq{}& \Delta^{(4)}_{abcd} + \frac{1}{72}(\omega_{;c}\omega_{;d} + \omega_{;cd})R_{ab} + \frac{1}{120}(7\omega_{;ab}\omega_{;cd} + 6\omega_{;a}\omega_{;bcd}) + \frac{1}{120}(\omega_{;a}\omega_{;b}\omega_{;c}\omega_{;d} + 2\omega_{;a}\omega_{;b}\omega_{;cd} + 3\omega_{;abcd})\\ &+ \frac{g_{ab}}{720}\{6\omega_{;c}(\omega_{;d}\omega_{;p}{}^{p} - \omega_{;p}{}^{p}{}_{d}) + 3(4\omega_{;p}{}^{p}\omega_{;cd} - 12\omega_{;pc}\omega_{;d}{}^{p} + 3\omega_{;p}{}^{p}{}_{cd}) - 12\omega_{;p}(\omega_{;c}\omega_{;d}\omega^{;p} + 2\omega^{;p}\omega_{;cd} - 2\omega_{;c}\omega_{;d}{}^{p}\\ &- \omega_{;cd}{}^{p} + 4\omega_{;c}{}^{p}{}_{d}) + 6\omega_{;p}(R_{cd}{}^{;p} - R_{c}{}^{p}{}_{;d}) - 5(2\omega_{;p}\omega^{;p} - \omega_{;p}{}^{p})R_{cd} - 2(2\omega_{;p}\omega_{;d} - \omega_{;pd})R_{c}{}^{p} + 4(\omega_{;p}\omega_{;q} + \omega_{;pq})R_{c}{}^{p}{}_{d}{}^{q}\}\\ &+ \frac{g_{ab}g_{cd}}{1440}\{2\omega_{;p}\omega_{;q}(6\omega^{;p}\omega^{;q} - 14\omega^{;pq} + 3R^{pq}) - 14\omega_{;p}\omega^{;p}\omega_{;q}{}^{q} + 5\omega_{;p}{}^{p}\omega_{;q}{}^{q} + 4\omega_{;pq}\omega^{;pq} + 12\omega_{;p}\omega_{;q}{}^{qp}\}.\end{aligned} \tag{F16c}$$

The $\overline{\Delta}^{(5)}_{abcde}$ and $\overline{\Delta}^{(6)}_{abcdef}$ expressions can be found in Ref. [48].

Next, we present the explicit expressions for the expansion tensors for $G_{\text{div,ren}}$. The expansion scalars in the body are related to these tensors via $G^{(n)}_{\text{div,ren}} = 2^{n/2}p^{a_1}\cdots p^{a_n}G^{(n)}_{\text{div,ren},a_1\cdots a_n}$:

$$G^{(0)}_{\text{div,ren}} = \frac{1}{6}(\omega_{;p}{}^{p} - \omega_{;p}\omega^{;p}), \tag{F17a}$$

$$G^{(1)}_{\text{div,ren},a} = \frac{1}{12}(2\omega_{;p}\omega_{;a}{}^{p} - \omega_{;p}{}^{p}{}_{a}), \tag{F17b}$$

$$\begin{aligned}G^{(2)}_{\text{div,ren},ab} ={}& (4\omega_{;p}\omega_{;a}\omega_{;b}\omega^{;p} + 15\omega_{;p}R_{ab}{}^{;p} - 15\omega_{;p}R_{a}{}^{p}{}_{;b} - 4\omega_{;a}\omega_{;b}\omega_{;p}{}^{p} + 2\omega_{;p}{}^{p}\omega_{;ab} + 8\omega_{;p}\omega_{;a}\omega_{;b}{}^{p} - 20\omega_{;pa}\omega_{;b}{}^{p}\\ &+ 6\omega_{;a}\omega_{;pb}{}^{p} - 12\omega_{;a}\omega_{;p}{}^{p}{}_{b} + 36\omega_{;p}\omega_{;ab}{}^{p} - 54\omega_{;p}\omega_{;a}{}^{p}{}_{b} - 12\omega_{;pab}{}^{p} + 3\omega_{;pa}{}^{p}{}_{b} + 9\omega_{;p}{}^{p}{}_{ab} + 9\omega_{;abp}{}^{p} - 5\omega_{;p}\omega^{;p}R_{ab}\\ &+ 5\omega_{;p}{}^{p}R_{ab} - 10\omega_{;p}\omega_{;a}R_{b}{}^{p} + 5\omega_{;pa}R_{b}{}^{p} + 34\omega_{;p}\omega_{;q}R_{a}{}^{q}{}_{b}{}^{p} + 10\omega_{;pq}R_{a}{}^{q}{}_{b}{}^{p})/360 - g_{ab}(2\omega_{;p}\omega_{;q}\omega^{;p}\omega^{;q}\\ &+ 4\omega_{;p}\omega^{;p}\omega_{;q}{}^{q} - 5\omega_{;p}{}^{p}\omega_{;q}{}^{q} + 16\omega_{;p}\omega_{;q}\omega^{;pq} - 4\omega_{;pq}\omega^{;pq} - 12\omega_{;p}\omega_{;q}{}^{qp} - 6\omega_{;p}\omega_{;q}R^{pq})/720.\end{aligned} \tag{F17c}$$

The expressions for $G^{(3)}_{\text{div,ren},abc}, G^{(4)}_{\text{div,ren},abcd}$ can be found in Ref. [48].

Finally we can present the explicit expressions for the expansion tensors for G_{fin}:

$$G^{(0)}_{\text{fin}} = \frac{\kappa^2}{6e^{2w}}, \tag{F18a}$$

$$G^{(1)}_{\text{fin},a} = \frac{\kappa^2 w_{;a}}{6e^{2w}}, \tag{F18b}$$

$$G^{(2)}_{\text{fin},ab} = \frac{\kappa^2}{72e^{2w}}(8w_{;a}w_{;b} - 4w_{;ab} - 2w_{;p}w^{;p}g_{ab} + w_{;p}{}^{p}g_{ab} + R_{ab}) + \frac{\kappa^4}{360e^{4w}}(4e^{2w}\delta^{\tau}_{a}\delta^{\tau}_{b} - g_{ab}), \tag{F18c}$$

$$\begin{aligned}G^{(3)}_{\text{fin},abc} ={}& \frac{\kappa^2}{144e^{2w}}(8w_{;a}w_{;b}w_{;c} - R_{ab;c} - 12w_{;a}w_{;bc} + 2w_{;abc} - 4w_{;p}w_{;a}w^{;p}g_{bc} + 2w_{;a}w_{;p}{}^{p}g_{bc} + 4w_{;p}w_{;a}{}^{p}g_{bc} - w_{;pa}{}^{p}g_{bc}\\ &+ 2w_{;a}R_{bc}) + \frac{\kappa^4}{180e^{4w}}(2e^{2w}\Gamma^{\tau}{}_{ab}\delta^{\tau}_{c} + 2e^{2w}w_{;a}\delta^{\tau}_{b}\delta^{\tau}_{c} - w_{;a}g_{bc}),\end{aligned} \tag{F18d}$$

$$
\begin{aligned}
G^{(4)}_{\mathrm{fin},abcd} = {} & \frac{\kappa^2}{8640 e^{2w}} \big(192 w_{;a} w_{;b} w_{;c} w_{;d} - 60 w_{;a} R_{bc;d} - 576 w_{;a} w_{;b} w_{;cd} + 144 w_{;ab} w_{;cd} + 18 R_{ab;cd} + 192 w_{;a} w_{;bcd} - 24 w_{;abcd} \\
& + 80 w_{;a} w_{;b} R_{cd} - 40 w_{;ab} R_{cd} + 5 R_{ab} R_{cd} + 4 R_{paqb} R_c{}^p{}_d{}^q - 2 g_{ab} \{ 72 w_{;p} w_{;c} w_{;d} w^{;p} - 15 w_{;p} R_{cd}{}^{;p} + 15 w_{;p} R_c{}^p{}_{;d} \\
& - 36 w_{;c} w_{;d} w_{;p}{}^p - 36 w_{;p} w^{;p} w_{;cd} + 18 w_{;p}{}^p w_{;cd} - 144 w_{;p} w_{;c} w_{;d}{}^p + 36 w_{;pc} w_{;d}{}^p - 6 w_{;c} w_{;pd}{}^p + 42 w_{;c} w_{;p}{}^p{}_d \\
& - 18 w_{;p} w_{;cd}{}^p + 54 w_{;p} w_{;cd}{}^p + 12 w_{;pcd}{}^p - 3 w_{;pcd}{}^p - 9 w_{;pcd}{}^p - 9 w_{;cdp}{}^p + 10 w_{;p} w^{;p} R_{cd} - 5 w_{;p}{}^p R_{cd} + 10 w_{;p} w_{;c} R_d{}^p \\
& - 5 w_{;pc} R_d{}^p - 10 w_{;p} w_{;q} R_c{}^p{}_d{}^q - 10 w_{;pq} R_c{}^q{}_d{}^p \} + g_{ab} g_{cd} \{ 12 w_{;p} w_{;q} w^{;p} w^{;q} - 14 w_{;p} w^{;p} w_{;q}{}^q + 5 w_{;p}{}^p w_{;q}{}^q \\
& - 28 w_{;p} w_{;q} w^{;pq} + 4 w_{;pq} w^{;pq} + 12 w_{;p} w_{;q}{}^{qp} 6 w_{;p} w_{;q} R^{pq} \} \big) + \frac{\kappa^4}{4320 e^{4w}} \big(4 e^{2w} \{ 3 \Gamma^\tau{}_{ab} \Gamma^\tau{}_{cd} + 12 \Gamma^\tau{}_{ab} w_{;c} \delta^\tau_d - 4 \Gamma^\tau{}_{ab;c} \delta^\tau_d \\
& + 8 w_{;a} w_{;b} \delta^\tau_c \delta^\tau_d - 4 w_{;ab} \delta^\tau_c \delta^\tau_d + \delta^\tau_c \delta^\tau_d R_{ab} \} - g_{ab} \{ 28 w_{;c} w_{;d} - 8 w_{;cd} + R_{cd} + 4 e^{2w} [2 w_{;p} w^{;p} - w_{;p}{}^p] \delta^\tau_c \delta^\tau_d R_{cd} \} \\
& + g_{ab} g_{cd} \{ 3 w_{;p} w^{;p} - w_{;p}{}^p \} \big) + \frac{\kappa^6}{15120 e^{6w}} \big(16 e^{4w} \delta^\tau_a \delta^\tau_b \delta^\tau_c \delta^\tau_d - 12 e^{2w} \delta^\tau_c \delta^\tau_d g_{ab} + g_{ab} g_{cd} \big). \qquad \text{(F18e)}
\end{aligned}
$$

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